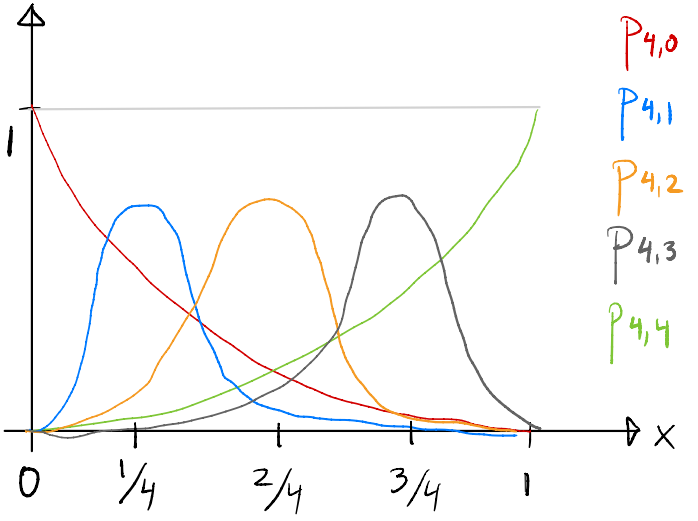
WEIERSTRASS' APPROXIMATION THEOREM

II

Def: The **Bernstein polynomials** are

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad \text{for } x \in [0,1]$$

for $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n$.



Facts:

- $p_{n,k}(x) \in [0,1] \quad \forall \quad x \in [0,1]$
- $p_{n,k}$ has a single max. at $x_{n,k} = k/n$
- $\sum_{k=0}^{n} p_{n,k}(x) = 1 \quad \forall \quad x \in [0,1]$
- $\sum_{k=0}^{n} p_{n,k}(x)\, k = nx \quad \forall \quad x \in [0,1]$
- $\sum_{k=0}^{n} p_{n,k}(x)\, k^2 = xn + x^2 n(n-1) \quad \forall \quad x \in [0,1]$

Def:

Let $f : [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$. The *n-th Bernstein approximation* of f is $f_n \in \mathbb{P}_n$ given by

$$f_n(x) = \sum_{k=0}^{n} p_{n,k}(x)\, f(x_{n,k})$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

Weierstrass' approximation theorem: Let $a < b$. For every $f \in C([a,b])$ there is a sequence $\{f_n\}_n$ of polynomials such that $f_n \xrightarrow[n\to\infty]{} f$ uniformly.

Proof: We first transform $[a,b]$ to $[0,1]$, and let $f \in C([0,1])$. Since f is continuous on a compact set, it is uniformly continuous, so if $\varepsilon > 0$ then there is some $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{for all } x, y \in [0,1] \text{ s.t. } |x-y| < \delta.$$

Let $M = \sup_{x\in[0,1]} |f(x)|$.

If f_n is the n-th Bernstein approx. of f then

$$|f_n(x) - f(x)| = \left| \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right) - \underbrace{\sum_{k=0}^{n} p_{n,k}(x)}_{=1} f(x) \right|$$

$$\leq \sum_{k=0}^{n} p_{n,k}(x) \left| f\left(\frac{k}{n}\right) - f(x) \right|$$

$$= \sum_{k:\left|\frac{k}{n}-x\right|<\delta} p_{n,k}(x) \underbrace{\left| f\left(\frac{k}{n}\right) - f(x) \right|}_{\leq \varepsilon} + \sum_{k:\left|\frac{k}{n}-x\right|\geq\delta} p_{n,k}(x) \underbrace{\left| f\left(\frac{k}{n}\right) - f(x) \right|}_{\leq 2M}$$

$$\leq \sum_{k=0}^{n} p_{n,k}(x) \cdot \varepsilon + \sum_{k:\left|\frac{k}{n}-x\right|\geq\delta} p_{n,k}(x) \cdot 2M$$

$$= \varepsilon + 2M \sum_{k:\left|\frac{k}{n}-x\right|\geq\delta} p_{n,k}(x)$$

$$\Rightarrow \quad |f_n(x) - f(x)| \leq \varepsilon + 2M \sum_{k : \left|\frac{k}{n} - x\right| \geq \delta} p_{n,k}(x).$$

If $\left|\frac{k}{n} - x\right| \geq \delta$ then $1 \leq \frac{\left(\frac{k}{n} - x\right)^2}{\delta^2} = \frac{k^2}{n^2\delta^2} - 2\frac{kx}{n\delta^2} + \frac{x^2}{\delta^2}$

so

$$|f_n(x) - f(x)| \leq \varepsilon + \frac{2M}{\delta^2}\left(\frac{1}{n^2}\sum_{k=0}^{n} p_{n,k}(x)k^2 - \frac{2x}{n}\sum_{k=0}^{n} p_{n,k}(x)\cdot k + x^2 \sum_{k=0}^{n} p_{k,n}(x)\right)$$

$$= \varepsilon + \frac{2M}{\delta^2}\left(\frac{1}{n^2}\left(xn + x^2 n(n-1)\right) - \frac{2x}{n}nx + x^2\right)$$

$$= \varepsilon + \frac{2Mx(1-x)}{n\delta^2}$$

$$\leq \varepsilon + \frac{2M}{n\delta^2}$$

We have shown that $|f_n(x) - f(x)| \leq \varepsilon + \frac{2M}{n\delta^2}$.

Now choose n so that $\frac{2M}{n\delta^2} \leq \varepsilon \iff n \geq \frac{2M}{\varepsilon\delta^2}$

Then $$|f_n(x) - f(x)| \leq 2\varepsilon \qquad \forall \quad x \in [0,1]$$

$$\Rightarrow \qquad \rho(f_n, f) \leq 2\varepsilon \qquad \forall \quad n \geq \frac{2M}{\varepsilon\delta}$$

so $f_n \xrightarrow[n \to \infty]{} f$ uniformly.



Note: The main trick in the proof was the estimate

$$\sum_{k:\,\left|\frac{k}{n}-x\right|\geq\delta} p_{n,k}(x) \leq \sum_{k} \frac{\left(\frac{k}{n}-x\right)^2}{\delta^2}\, p_{n,k}(x).$$

This is a variant of Chebyshev's inequality: if $A=\{z\in X : g(z)\geq\alpha\}$ and $f\geq 0$ then

$$\int_A f(z)\,dz \leq \int_X \frac{|g(z)|^r}{|\alpha|^r} f(z)\,dz \qquad \text{for any } r>0.$$

Corollary

The set of polynomials $\mathbb{P}$ is dense in $C([a,b])$.

Recall: D is dense in X if for all $x \in X$ there are $d_1, d_2, \ldots \in D$ so that $d_n \xrightarrow[n \to \infty]{} x$.

Corollary

$C([a,b])$ is separable.

Proof: For every $n \in \mathbb{N}$, the set

$$\mathbb{P}_{\mathbb{Q},n} = \{ p : \mathbb{R} \to \mathbb{R} \ : \ p(x) = a_0 + a_1 x + \cdots + a_n x^n \ \forall\, x \in \mathbb{R} \text{ for } a_0, \ldots, a_n \in \mathbb{Q} \}$$

is countable. Hence, $\mathbb{P}_{\mathbb{Q}} = \bigcup_{n \in \mathbb{N}_0} \mathbb{P}_{\mathbb{Q},n}$ is countable.

But $\mathbb{P}_{\mathbb{Q}}$ is dense in $\mathbb{P}$, which is dense in $C([a,b])$, so $\mathbb{P}_{\mathbb{Q}}$ is dense in $C([a,b])$. ∎

QUESTIONS?

COMMENTS?