# MODES OF CONVERGENCE

Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces.

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions $f_n : X \to Y$ and $f : X \to Y$ another function. We say that

- $\{f_n\}_n$ converges to $f$ pointwise if $\forall\ \varepsilon > 0$ and $x \in X$ $\exists\ N \in \mathbb{N}$ such that $d_Y(f_n(x), f(x)) < \varepsilon \quad \forall\ n \geq N$.
  We write $f_n \xrightarrow[n\to\infty]{} f$ pointwise.
- $\{f_n\}_n$ converges to $f$ uniformly if $\forall\ \varepsilon > 0$ $\exists\ N \in \mathbb{N}$ such that $d_Y(f_n(x), f(x)) < \varepsilon \quad \forall\ n \geq N$ and all $x \in X$.
  We write $f_n \xrightarrow[n\to\infty]{} f$ uniformly.

Uniform convergence $\Longrightarrow$ pointwise convergence

Let $\{f_n\}_n$ be a sequence of functions $f_n : X \to Y$, and let $f : X \to Y$. Then TFAE:

(i) $f_n \xrightarrow[n\to\infty]{} f$ pointwise

(ii) $f_n(x) \xrightarrow[n\to\infty]{} f(x)$ for every $x \in X$.

For $f, g : X \to Y$, define the <u>supremum metric</u>

$$\rho(f,g) = \sup_{x \in X} d_Y(f(x), g(x))$$

(Also denoted $d_\infty(f,g)$, $\|f-g\|_\infty$ or $\|f-g\|_{L^\infty}$.)

Let $\{f_n\}_n$ be a sequence of functions $f_n : X \to Y$, and let $f : X \to Y$. Then TFAE:

(i) $f_n \xrightarrow[n\to\infty]{} f$ uniformly

(ii) $\rho(f_n, f) \xrightarrow[n\to\infty]{} 0$

Proof: (i) $\iff$ $\forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N}$: $d_Y(f_n(x), f(x)) \leq \varepsilon$ $\forall\, n \geq N$, $\forall\, x \in X$

$\iff$ $\forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N}$: $\sup_{x \in X} d_Y(f_n(x), f(x)) \leq \varepsilon$ $\forall\, n \geq N$

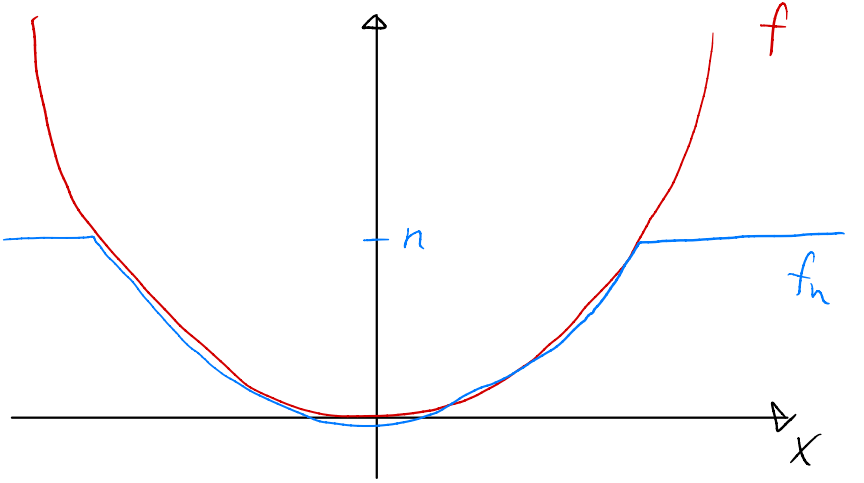
$\iff$ $\forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N}$: $\rho(f_n, f) \leq \varepsilon$ $\forall\, n \geq N$

$\iff$ $\rho(f_n, f) \xrightarrow[n\to\infty]{} 0$.

∎

**Example:** Let $f_n, f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, $f_n(x) = \min(n, x^2)$.
Show that $f_n \xrightarrow[n\to\infty]{} f$ pointwise, but not uniformly

For any $x \in \mathbb{R}$, we have
$f_n(x) = f(x)$ for every $n \geq x^2$,
so $f_n(x) \xrightarrow[n\to\infty]{} f(x)$.



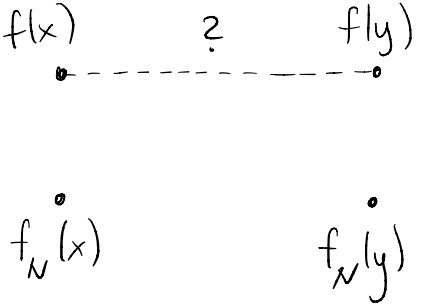

Next,

$$\rho(f_n, f) = \sup_{x\in\mathbb{R}} |f_n(x) - f(x)|$$

$$= \sup_{x\in\mathbb{R}} |\min(n, x^2) - x^2| = \infty$$

$$\not\xrightarrow[n\to\infty]{} 0, \quad \text{so} \quad f_n \not\xrightarrow[n\to\infty]{} f \text{ uniformly.}$$

Let $\{f_n\}_n$ be a sequence of continuous functions $f_n : X \to Y$, and assume $f_n \xrightarrow[n\to\infty]{} f$ uniformly for some $f : X \to Y$. Then $f$ is continuous.




Proof: Let $x \in X$ and $\varepsilon > 0$.

Let $N \in \mathbb{N}$ be such that

$$\rho(f_n, f) < \varepsilon \quad \forall\, n \geq N.$$

Let $\delta > 0$ be such that

$d_Y(f_N(x), f_N(y)) < \varepsilon$ $\forall$ $y$ such that $d_X(x, y) < \delta$. Then

$$d_Y(f(x), f(y)) \leq d_Y(f(x), f_N(x)) + d_Y(f_N(x), f_N(y)) + d_Y(f_N(y), f(y)) < 3\varepsilon.$$



Let $\{f_n\}_n$ be a sequence of uniformly continuous functions $f_n : X \to Y$, and assume $f_n \xrightarrow[n\to\infty]{} f$ uniformly for some $f : X \to Y$. Then $f$ is uniformly continuous.

To summarize:

- $f_n \xrightarrow[n\to\infty]{} f$ pointwise $\iff$ $f_n(x) \xrightarrow[n\to\infty]{} f(x)$ $\forall x \in X$
- $f_n \xrightarrow[n\to\infty]{}$ uniformly $\iff$ $\rho(f_n, f) \xrightarrow[n\to\infty]{} 0$
- the uniform limit of continuous functions is continuous
- the uniform limit of uniformly continuous functions is uniformly continuous.

QUESTIONS?

COMMENTS?