DIFFERENTIATING SEQUENCES OF FUNCTIONS

For functions $f_1, f_2, \ldots : [a, b] \longrightarrow \mathbb{R}$

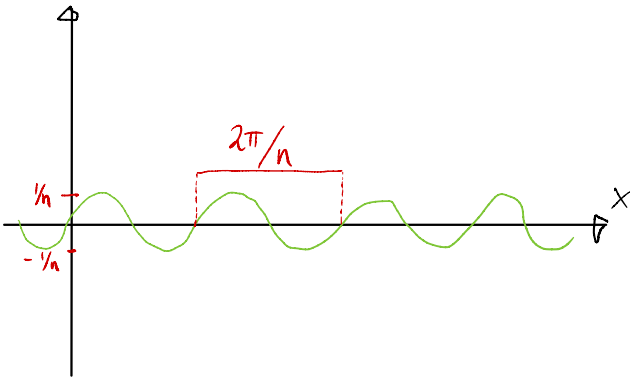
$$\lim_{n\to\infty} \int_a^x f_n(t)\,dt = \int_a^x \lim_{n\to\infty} f_n(t)\,dt$$

whenever the limit $\lim_{n\to\infty} f_n$ is uniform.

What about $\lim_{n\to\infty} \left(\frac{d}{dx} f_n(x)\right) = \frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right)$?

It's not enough that $\{f_n\}_n$ converges uniformly.

Example: Let $f_n(x) = \frac{\sin(nx)}{n}$.



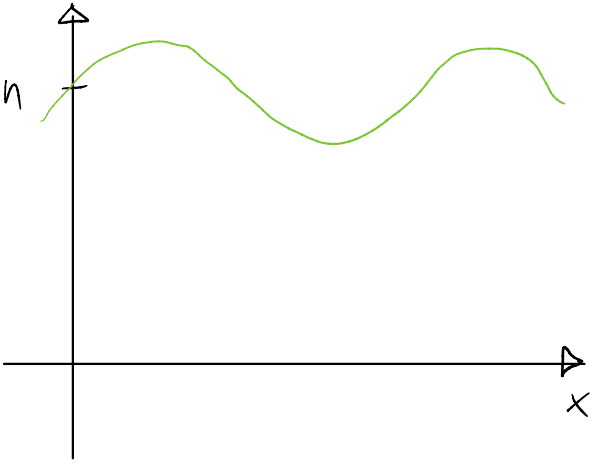
Then $|f_n(x)| \leq \frac{1}{n} \xrightarrow[n\to\infty]{} 0$,

so $f_n \xrightarrow[n\to\infty]{} 0$ uniformly,

but $\frac{d}{dx} f_n(x) = \cos(nx)$ does not converge.

It's not enough that $\{f_n'\}_n$ converges.

Example: Let $f_n(x) = \sin(x) + n$.



Then $f_n'(x) = \cos(x)$, so $f_n' \xrightarrow[n\to\infty]{} \cos$ uniformly, but $\{f_n\}_n$ does not converge!

Let $f_n : [a,b] \to \mathbb{R}$ be continuously differentiable $\forall\, n \in \mathbb{N}$. Assume:

- $\{f_n'\}_{n\in\mathbb{N}}$ converges uniformly to some $g : [a,b] \to \mathbb{R}$
- $\{f_n(x_0)\}_{n\in\mathbb{N}}$ converges to some $\alpha \in \mathbb{R}$, for some $x_0 \in \mathbb{R}$.

Then $f_n \xrightarrow[n\to\infty]{} f$ unif. for some $f : [a,b] \to \mathbb{R}$ with $f' = g$ and $f(x_0) = \alpha$.

Proof: Let $g_n = f_n'$. Then $g_n \xrightarrow[n\to\infty]{} g$ uniformly, and

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} g_n(t)\,dt.$$

Now apply the result about integrating limits to get $f_n \xrightarrow[n\to\infty]{} f$ uniformly, where

$$f(x) = \lim_{n\to\infty} f_n(x_0) + \lim_{n\to\infty} \int_{x_0}^{x} g_n(t)\,dt = \alpha + \int_{x_0}^{x} g(t)\,dt.$$

∎

If $\{u_n\}_n$ is a sequence of continuously differentiable functions $u_n : [a,b] \longrightarrow \mathbb{R}$ such that $\sum_{n=1}^{\infty} u_n'$ converges uniformly and $\sum_{n=1}^{\infty} u_n(x_0)$ converges for some $x_0 \in [a,b]$, then $\sum_{n=1}^{\infty} u_n$ converges uniformly, and $\frac{d}{dx}\left(\sum_{n=1}^{\infty} u_n\right) = \sum_{n=1}^{\infty}\left(\frac{d}{dx} u_n\right)$.

Proof: Apply the previous result to $f_n(x) = \sum_{k=1}^{n} u_n(x)$. ∎

QUESTIONS?

COMMENTS?