INTEGRATING SEQUENCES OF FUNCTIONS

Let $\{f_n\}_n$ be a convergent sequence of functions

$$f_n : [a,b] \longrightarrow \mathbb{R}.$$

Is $$\lim_{n\to+\infty} \int_a^b f_n(t)\,dt = \int_a^b \lim_{n\to+\infty} f_n(t)\,dt \quad ?$$

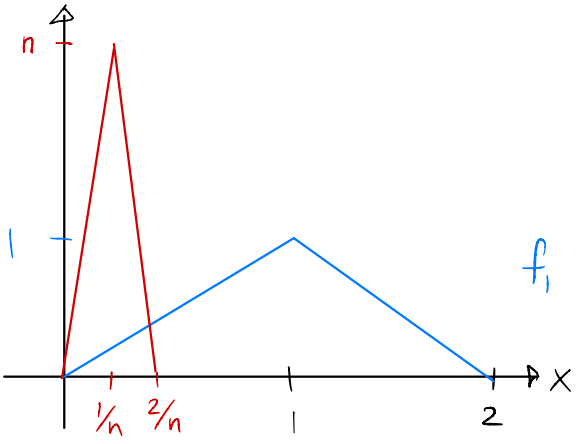
Is $$\lim_{n\to+\infty} \left(\frac{d}{dx} f_n(x)\right) = \frac{d}{dx}\left(\lim_{n\to+\infty} f_n(x)\right) \quad ?$$

Example: Let $f_1(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 < x \le 2 \\ 0 & \text{else,} \end{cases}$ $f_n(x) = n f_1(nx)$

for $x \in [0,2]$.

Then $f_n \xrightarrow[n\to\infty]{} 0$ pointwise:

- If $x \le 0$ then $f_n(x) = 0 \quad \forall n$
- If $x > 0$ then $f_n(x) = 0 \quad \forall n > \frac{2}{x}$



But

$$\lim_{n\to\infty} \int_0^2 f_n(t)\,dt = 1 \neq \int_0^2 \lim_{n\to\infty} f_n(t)\,dt = 0.$$

Let $f_n : [a,b] \to \mathbb{R}$ be continuous $\forall\, n \in \mathbb{N}$ and assume $f_n \xrightarrow[n\to\infty]{} f$ uniformly. Let $F_n(x) = \int_a^x f_n(t)\,dt$, $F(x) = \int_a^x f(t)\,dt$.

Then $F_n \xrightarrow[n\to\infty]{} F$ uniformly.

In other words: $\lim_{n\to\infty} \int_a^x f_n(t)\,dt = \int_a^x \lim_{n\to\infty} f_n(t)\,dt$.

Proof: Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that $\rho(f_n, f) < \varepsilon$ when $n \geq N$. Then

$$|F_n(x) - F(x)| = \left|\int_a^x f_n(t) - f(t)\,dt\right| \leq \int_a^x |f_n(t) - f(t)|\,dt$$

$$\leq \int_a^x \rho(f_n, f)\,dt = (x-a)\,\rho(f_n, f) \leq (b-a)\,\varepsilon.$$

∎

Corollary

Let $f_n : [a,b] \to \mathbb{R}$ be continuous $\forall\, n \in \mathbb{N}$ and assume $f_n \xrightarrow[n\to\infty]{} f$ uniformly. Let $F_n(x) = \int_{x_0}^{x} f_n(t)\,dt$, $F(x) = \int_{x_0}^{x} f(t)\,dt$ for some $x_0 \in [a,b]$. Then $F_n \xrightarrow[n\to\infty]{} F$ uniformly.

The functions f_n must be defined on a closed, bounded interval!

Example: Let $f_1(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 < x \le 2 \\ 0 & \text{else,} \end{cases}$ $f_n(x) = \frac{f(x/n)}{n}$.

for $x \in [0, \infty)$.

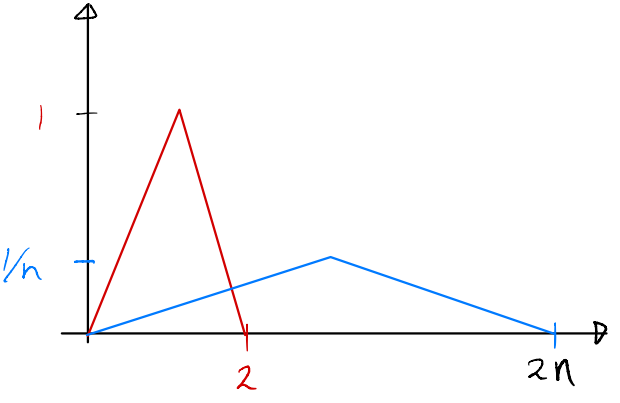
Then $f_n \xrightarrow[n\to\infty]{} 0$ uniformly, since

$\rho(f_n, 0) = 1/n \xrightarrow[n\to\infty]{} 0$,

but

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = 1 \neq \int_0^\infty \lim_{n\to\infty} f_n(x)\,dx = 0$$

($\int_0^\infty f_n(x)\,dx = 1$; $\lim_{n\to\infty} f_n(x) = 0$)



Series of functions

Given $v_1, v_2, \ldots : [a,b] \to \mathbb{R}$, what do we mean by

$$\sum_{n=1}^{\infty} v_n(x) \quad ?$$

Recall: If $a_1, a_2, \ldots$ are numbers then we say that the series $\sum_{n=1}^{\infty} a_n$ converges if the partial sums $S_N = \sum_{n=1}^{N} a_n$ converge as $N \to \infty$.

Let (X, d) be a metric space.

Let $A \subseteq X$ and let $\{v_n\}_{n \in \mathbb{N}}$ be a sequence of functions $v_n : A \to \mathbb{R}$. We say that the series $\sum_{n=1}^{\infty} v_n$

- converges pointwise if for every $x \in A$, the partial sums $S_N(x) = \sum_{n=1}^{N} v_n(x)$ converge as $N \to \infty$.
- converges uniformly if the partial sums S_N converge uniformly as $N \to \infty$.

Weierstrass' M-test

Assume that there are numbers $M_n \geq 0$ such that

- $|v_n(x)| \leq M_n \quad \forall \; x \in A$
- $\sum_{n=1}^{\infty} M_n < \infty$.

Then $\sum_{n=1}^{\infty} v_n$ converges uniformly.

Proof: Let $x \in A$. Then $\sum_{n=1}^{\infty} |v_n(x)| \leq \sum_{n=1}^{\infty} M_n < \infty$, so the series $\sum_{n=1}^{\infty} v_n(x)$ converges absolutely. Hence, the series converges; let $s(x) = \sum_{n=1}^{\infty} v_n(x)$. Now, $\forall \; x \in A$,

$$\left| \sum_{n=1}^{N} v_n(x) - s(x) \right| \leq \sum_{n=N+1}^{\infty} |v_n(x)| \leq \sum_{n=N+1}^{\infty} M_n \xrightarrow[N \to \infty]{} 0.$$



Let $v_n : [a,b] \to \mathbb{R}$ be continuous $\forall\, n \in \mathbb{N}$ and assume $\sum_{n=1}^{\infty} v_n$ converges uniformly. Let $V_n(x) = \int_a^x v_n(t)\,dt$. Then the series $\sum_{n=1}^{\infty} V_n$ converges uniformly, and

$$\sum_{n=1}^{\infty} V_n(x) = \int_a^x \sum_{n=1}^{\infty} v_n(t)\,dt \qquad \forall\ x \in [a,b].$$

Proof: Let $s_n(x) = \sum_{k=1}^{n} v_n(x)$. Then $\{s_n\}_n$ converges uniformly, so by our earlier result,

$$\int_a^x \sum_{n=1}^{\infty} v_n(t)\,dt = \int_a^x \lim_{n\to\infty} s_n(t)\,dt = \lim_{n\to\infty} \int_a^x s_n(t)\,dt = \lim_{n\to\infty} \sum_{k=1}^{n} \int_a^x v_n(t)\,dt.$$



QUESTIONS ?

COMMENTS ?

Next video: Differentiating sequences/series of functions