POWER SERIES

A **power series** is an expression of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where $a \in \mathbb{R}$ and $c_0, c_1, \ldots \in \mathbb{R}$ are fixed, and x takes values in $\mathbb{R}$. (NO: Potensrekke)

Examples:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad -\sum_{n=1}^{\infty} \frac{(1-x)^n}{n} \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

A function $f: B \to \mathbb{R}$ (for some open $B \subseteq \mathbb{R}$) is <u>real analytic</u> if for every $a \in B$ there are $R > 0$ and $c_0, c_1, \ldots \in \mathbb{R}$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad \forall\, x \in (a-R, a+R)$$

<u>Note</u>: The power series must converge at all $x \in (a-R, a+R)$!

<u>Examples</u>:

- $f(x) = e^x$
- $f = \cos$
- $f = \sin$
- $f = \tan$
- $f = \log$
- any polynomial

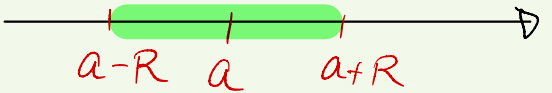
The radius of convergence of a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ is

$$R = \frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}} = \liminf_{n\to\infty} |c_n|^{-1/n}$$

Note: We use the convention $1/0 = \infty$ and $1/\infty = 0$.

Proposition: The power series converges if $|x-a| < R$, and diverges if $|x-a| > R$. It converges uniformly on every interval $[a-r, a+r]$ for $r < R$.

The power series converges if $|x-a|<R$, and diverges if $|x-a|>R$. It converges uniformly on every interval $[a-r, a+r]$ for $r<R$.



Proof: Recall that $$R = \frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}}.$$

If $|x-a|>R$ then $\frac{1}{|x-a|} < \frac{1}{R}$, so for any $N \in \mathbb{N}$ there is an $n \geq N$ such that $\frac{1}{|x-a|} < |c_n|^{1/n}$. But then $|c_n(x-a)^n| > 1$. Hence, $\sum_{n=0}^{\infty} c_n (x-a)^n$ cannot converge.

The power series converges if $|x-a|<R$, and diverges if $|x-a|>R$. It converges uniformly on every interval $[a-r, a+r]$ for $r<R$.

Proof: Next, if $R=0$ there is nothing more to prove. If $R>0$, let $r<R$. Then $\frac{1}{r} > \frac{1}{R} = \limsup_{n\to\infty} |c_n|^{1/n}$, so for some $N\in\mathbb{N}$ we have $|c_n|^{1/n} \le \eta \quad \forall\, n\ge N$, where $\eta = \frac{1}{2}\left(\frac{1}{r}+\frac{1}{R}\right)$. Note that $\eta r < \frac{1}{r} r = 1$. We have now

$$|c_n (x-a)^n| = |c_n|\,|x-a|^n \le \eta^n r^n = (\eta r)^n \qquad \forall\, n \ge N,$$

so $\sum_{n=0}^{\infty} |c_n (x-a)^n| \le \sum_{n=0}^{N-1} |c_n| r^n + \sum_{n=N}^{\infty} (\eta r)^n < \infty$. Thus, by Weierstrass' M-test, the series converges uniformly in $[a-r, a+r]$. ∎

Let R be the radius of convergence of the power series $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, and assume $R > 0$. Then f is continuous and differentiable on $(a-R, a+R)$, and

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} (x-a)^n .$$

Proof: Let $S_N(x) = \sum_{n=0}^{N} c_n (x-a)^n$. Then $S_N \xrightarrow[N\to\infty]{} f$ uniformly in $[a-r, a+r]$ for any $r < R$. Moreover,

$$S_N'(x) = \sum_{n=0}^{N} n c_n (x-a)^{n-1} = \sum_{n=0}^{N-1} (n+1) c_{n+1} (x-a)^n .$$ But

$$\limsup_{n\to\infty} |n c_n|^{1/n} = \limsup_{n\to\infty} \left(|n|^{1/n} |c_n|^{1/n} \right) = \left(\lim_{n\to\infty} n^{1/n} \right) \left(\limsup_{n\to\infty} |c_n|^{1/n} \right) = 1/R ,$$

so $\sum_{n=0}^{\infty} (n+1) c_{n+1} (x-a)^n$ has the same radius of convergence.

Let R be the radius of convergence of the power series $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, and assume $R > 0$. Then f is continuous and differentiable on $(a-R, a+R)$, and

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} (x-a)^n.$$

Proof (continued): It follows from our theorem on differentiating series that $f'(x) = \sum_{n=0}^{\infty} (n+1) c_{n+1} (x-a)^n$.



Let R be the radius of convergence of the power series $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, and assume $R > 0$. Then all derivatives of f exist, and $f^{(k)}(a) = k!\, c_k$, so

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n .$$

Proof: Apply the previous result k times to get differentiability and

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, c_n (x-a)^{n-k}$$

$$= \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\, c_n (x-a)^{n-k} .$$

Set $x = a$ to get $f^{(k)}(a) = \frac{k!}{0!} c_k = k!\, c_k$.



Thus, any real analytic function is equal to its own Taylor series.

QUESTIONS?

COMMENTS?