THE SPACE OF BOUNDED, CONTINUOUS FUNCTIONS

Let (X, d_X) and (Y, d_Y) be metric spaces.

Recall:
- $B(X, Y) = \{\text{all bounded } f : X \to Y\}$
- $(B(X,Y), \rho)$ is a metric space
- $(B(X,Y), \rho)$ is complete whenever (Y, d_Y) is.

Recall: $C(X,Y) = \{\text{all continuous } f : X \to Y\}$

Define $C_b(X,Y) = C(X,Y) \cap B(X,Y)$

$= \{\text{bounded, continuous functions } f : X \to Y\}$

From now on we will study the metric space

$$\left(C_b(X,Y), \rho\right)$$

Prop.: $C_b(X, Y)$ is a closed subset of $B(X, Y)$

Proof: If $\{f_n\}_n$ is a sequence in $C_b(X, Y)$ converging to $f \in B(X, Y)$ then f is continuous, since the uniform limit of continuous functions is continuous. Hence, $f \in C_b(X, Y)$.



Corollary: If (Y, d_Y) is complete then so is $(C_b(X,Y), \rho)$.

Proof: $C_b(X,Y)$ is a closed subset in the metric space $(B(X,Y), \rho)$, and the closed subset of a complete metric space is complete.



Proposition: If (X, d_X) is compact then $C(X,Y) = C_b(X,Y)$.

Proof: If (X, d_X) is compact then every continuous function $f: X \to \mathbb{R}$ is bounded, by the extremal value theorem.

Let $g \in C(X,Y)$, fix any $\bar{y} \in Y$, and let

$$f(x) = d_Y(\bar{y}, g(x)).$$

Then $f: X \to \mathbb{R}$ is continuous, hence also bounded, so g is bounded.



QUESTIONS?

COMMENTS?