ORDINARY DIFFERENTIAL EQUATIONS

An ordinary differential equation is an equation of the form $y'(t) = f(t, y(t))$

where $y : [0,\infty) \to \mathbb{R}$ (or $y : [0,\infty) \to \mathbb{R}^n$) is the unknown and $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ (or $[0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$) is a given function.

Examples:

$$y'(t) = 2y(t)$$

$$y'(t) = t^2 y(t)$$

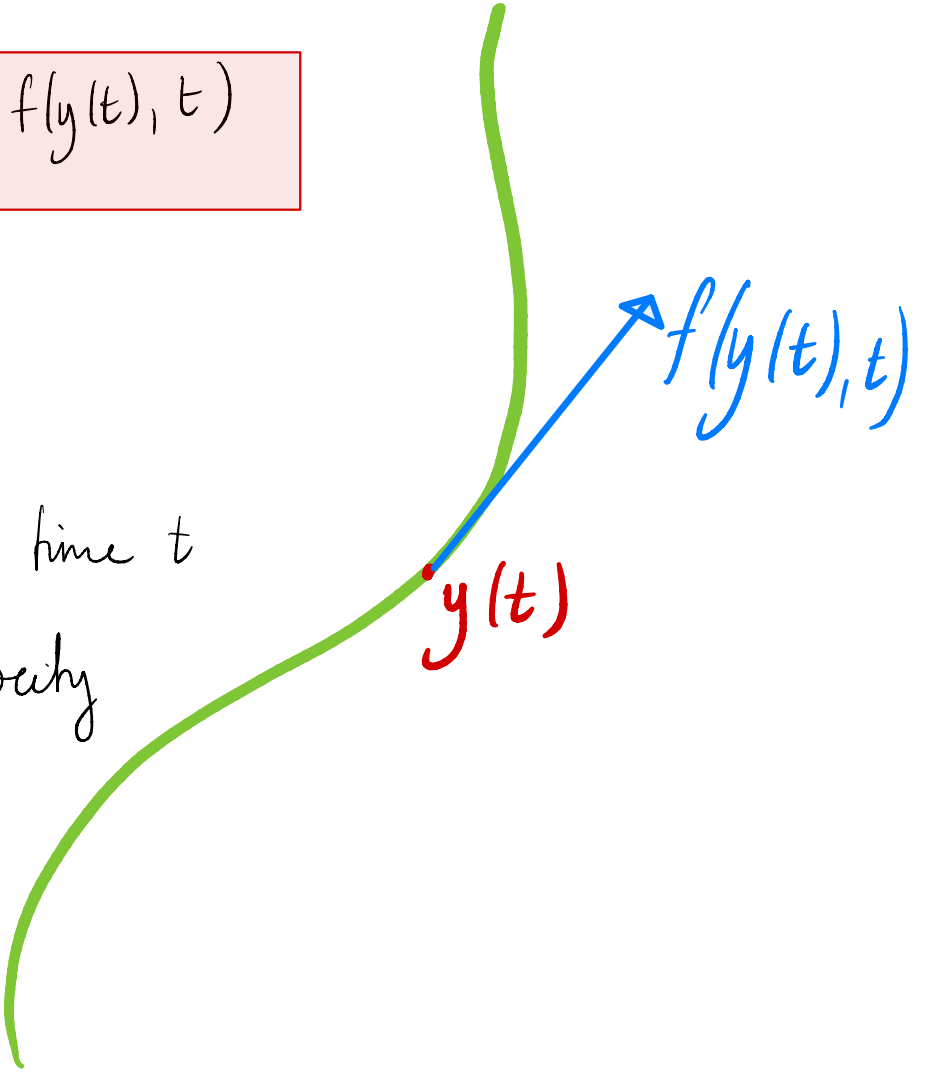
$$y'(t) = \cos(t + y(t))$$

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} y_2(t)^2 - t \\ y_1(t) + y_2(t) \end{pmatrix}$$

$$y'(t) = f(y(t), t)$$

It's useful to think of

- t as time
- $y(t)$ as position at time t
- $f(y(t), t)$ as the velocity



We supply the ODE with initial data:

$$(1)\quad \begin{cases} y'(t) = f(t, y(t)) & \text{for } t > 0 \\ y(0) = \bar{y} \end{cases}$$

for some given $\bar{y} \in \mathbb{R}$ (or $\bar{y} \in \mathbb{R}^n$).

Solving (1) means finding a function y satisfying (1).

$$(1)\quad \begin{cases} y'(t) = f(t, y(t)) & \text{for } t > 0 \\ y(0) = \bar{y} \end{cases}$$

We are only able to solve (1) for very specific choices of f (f linear / separable / ...) !

Can we still prove that a solution exists?

That it is unique?

That it is stable with respect to $\bar{y}$?

That we can approximate it?

Some simplifications

(Non-) autonomous ODEs

We can convert a non-autonomous ODE $y'(t) = f(t, y(t))$ into an autonomous ODE $z'(t) = g(z(t))$.

Let $z_1(t) = y_1(t), \dots, z_n(t) = y_n(t)$ and $z_{n+1}(t) = t$.

Let $g(z) = f(z_{n+1}, (z_1, \dots, z_n))$.

Then $y'(t) = f(t, y(t)) \iff z'(t) = g(z(t))$.

High-order ODE's

We can convert a high-order ODE into a first-order ODE

Consider $y''''(t) = y'''(t) + 5y''(t) - 3y'(t) - y(t)$.

Let $z_1 = y$, $z_2 = y'$, $z_3 = y''$, $z_4 = y'''$. Then

$$\begin{cases} z_1'(t) = z_2(t) \\ z_2'(t) = z_3(t) \\ z_3'(t) = z_4(t) \\ z_4'(t) = z_4(t) + 5z_3(t) - 3z_2(t) - z_1(t) \end{cases}$$

(and vice versa)

We will therefore only consider first-order, autonomous ODEs

$$\begin{cases} y'(t) = f(y(t)) & \text{for } t > 0 \\ y(0) = \overline{y} \end{cases}$$

keeping in mind that this covers a large class of ODEs.

For the sake of simplicity we will assume $n = 1$.

QUESTIONS?

COMMENTS?

Next video: (non-)uniqueness of solutions