UNIQUENESS FOR ODES

Consider (1) $\begin{cases} y'(t) = f(y(t)) & \text{for } t > 0 \\ y(0) = \overline{y} \end{cases}$

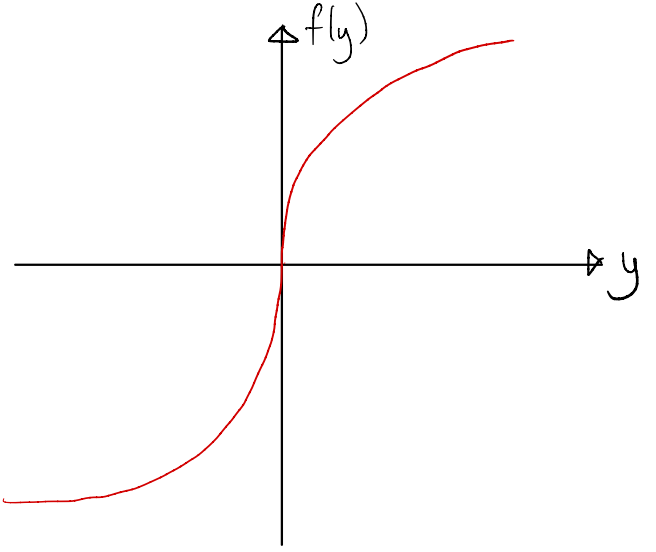
Is the solution unique ?

Without uniqueness, the model (1) lacks predictive power - it does not give a concrete prediction about the future.

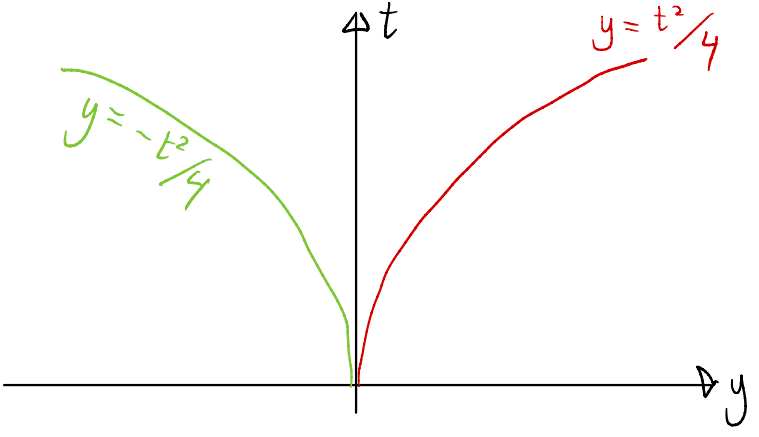
Non-uniqueness

Example: There are infinitely many solutions of

$$\begin{cases} y'(t) = \begin{cases} \sqrt{y(t)} & (y(t) \geq 0) \\ -\sqrt{-y(t)} & (y(t) < 0) \end{cases} \\ y(0) = 0 \end{cases}$$



All of $y(t) = t^2/4$, $y(t) = -t^2/4$ and $y(t) = 0$ are solutions.



Def: $f: \mathbb{R} \to \mathbb{R}$ is <u>Lipschitz</u> if there is some $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y| \qquad \forall\, x, y \in \mathbb{R}.$$

<u>Theorem</u>: Let $f: \mathbb{R} \to \mathbb{R}$ be Lipschitz. If $x, y: [0, \infty) \to \mathbb{R}$ solve

$$\begin{cases} x'(t) = f(x(t)) \\ x(0) = \overline{x} \end{cases} \quad \text{and} \quad \begin{cases} y'(t) = f(y(t)) \\ y(0) = \overline{y} \end{cases} \quad \text{for } \overline{x}, \overline{y} \in \mathbb{R},$$

then $|x(t) - y(t)| \leq |\overline{x} - \overline{y}|\, e^{Kt}$ for all $t > 0$.

In particular, if $\overline{x} = \overline{y}$ then $x = y$.

Solutions are <u>stable</u> with respect to the initial data.

Theorem: Let $f: \mathbb{R} \to \mathbb{R}$ be Lipschitz. If $x, y: [0,\infty) \to \mathbb{R}$ solve $\begin{cases} x'(t) = f(x(t)) \\ x(0) = \bar{x} \end{cases}$ and $\begin{cases} y'(t) = f(y(t)) \\ y(0) = \bar{y} \end{cases}$ for $x_0, y_0 \in \mathbb{R}$,

then $|x(t) - y(t)| \leq |\bar{x} - \bar{y}|\, e^{Kt}$ for all $t > 0$.

Proof: $\frac{d}{dt}(x-y)^2 = 2(x-y)(x'-y') = 2(x-y)(f(x)-f(y)) \leq 2|x-y||f(x)-f(y)|$

$\leq 2K|x-y|^2.$

Multiply by e^{-2Kt}: $0 \geq e^{-2Kt}\left(\frac{d}{dt}(x-y)^2 - 2K(x-y)^2\right) = \frac{d}{dt}\left(e^{-2Kt}(x-y)^2\right)$

so $e^{-2Kt}(x(t)-y(t))^2 \leq e^{-2K\cdot 0}(x(0)-y(0))^2 \iff |x(t)-y(t)| \leq |\bar{x}-\bar{y}|\, e^{Kt}.$



QUESTIONS?

COMMENTS?

Next video: Existence of solutions