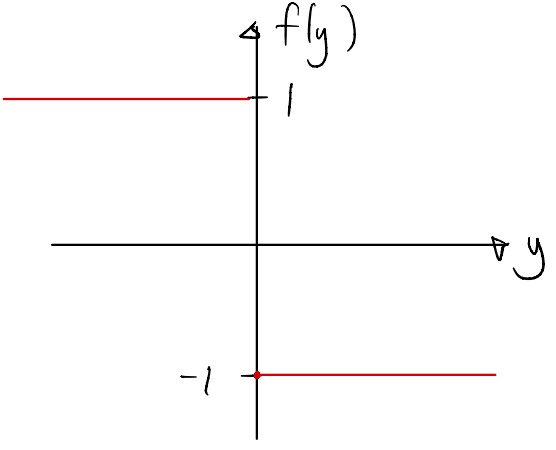
EXISTENCE FOR ODEs

Recall:

If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz then there is at most one solution of $\begin{cases} y'(t) = f(y(t)) & (t > 0) \\ y(0) = \bar{y} \end{cases}$ for any $\bar{y} \in \mathbb{R}$.

Non-existence



Example: There exists no solution of

$$\begin{cases} y'(t) = f(y(t)) & (t>0) \\ y(0) = 0 \end{cases} \quad \text{for } f(y) = \begin{cases} -1 & (y \geq 0) \\ 1 & (y<0) \end{cases}$$

A solution cannot increase from $y=0$ (since $y' = -1 < 0$),
cannot decrease from $y=0$ (since $y' = 1 > 0$),
and cannot stay at $y=0$ (since $y' = -1 \neq 0$).

Theorem (Peano): If f is continuous then there exists a solution.

Reformulation of the problem

Consider a solution of (1) $\begin{cases} y'(t) = f(y(t)) \\ y(0) = \bar{y} \end{cases}$ $(t > 0)$

Integrate over $[0, t]$:

$$y(t) - y(0) = \int_0^t y'(s)\,ds = \int_0^t f(y(s))\,ds$$

or:

$$(2) \quad y(t) = \bar{y} + \int_0^t f(y(s))\,ds \qquad \forall \quad t \geq 0$$

Conversely, if y satisfies (2) $\forall\ t \geq 0$, then (1) is satisfied.

We will use a fixed point iteration to solve (2)

Recall:

$f: X \to X$ is a contraction if there is some $K \in (0,1)$ such that $d(f(x), f(y)) \leq K\, d(x,y) \quad \forall\, x, y \in X$.

Banach's fixed point theorem

Let (X, d) be a complete metric space and $f: X \to X$ a contraction. Then f has a unique fixed point x.

Moreover, if $x_0 \in X$ and $x_{n+1} = f(x_n)$ $(n = 0, 1, 2, \ldots)$ then

$$x_n \xrightarrow[n \to \infty]{} x .$$

Theorem: (Cauchy – Lipschitz – Picard)
Assume that $f: \mathbb{R} \to \mathbb{R}$ is Lipschitz. Then there exists a unique solution of

$$\begin{cases} y'(t) = f(y(t)) & (t > 0) \\ y(0) = \overline{y} \end{cases}$$

Proof: We wish to find a solution of

$$(2) \qquad y(t) = \overline{y} + \int_0^t f(y(s))\, ds \qquad \forall \ t \geq 0$$

Let $T > 0$. For any $z \in C([0,T])$, let $F(z)$ be the function $F(z)(t) = \overline{y} + \int_0^t f(z(s))\, ds$.

Then (2) $\iff$ $F(y) = y$

$$F(z)(t) = \bar{y} + \int_0^t f(z(s))\,ds.$$

- Claim: $F(z) \in C([0,T])$ for all $z \in C([0,T])$.

  Indeed, $\bar{y}$ is a constant and $f \circ z$ is continuous, so $F(z)$ is even continuously differentiable.

$$F(z)(t) = \bar{y} + \int_0^t f(z(s))\,ds.$$

- **Claim:** $F : C([0,T]) \to C([0,T])$ is Lipschitz.

  If $z, \tilde{z} \in C([0,T])$ then

$$|F(z)(t) - F(\tilde{z})(t)| = \left| \int_0^t f(z(s)) - f(\tilde{z}(s))\,ds \right| \leq \int_0^t |f(z(s)) - f(\tilde{z}(s))|\,ds$$

$$\leq \int_0^t K \cdot |z(s) - \tilde{z}(s)|\,ds \leq Kt\,\rho(z,\tilde{z}).$$

$$\Rightarrow \quad \rho(F(z), F(\tilde{z})) \leq KT\,\rho(z,\tilde{z}).$$

Hence, F is Lipschitz with constant KT.

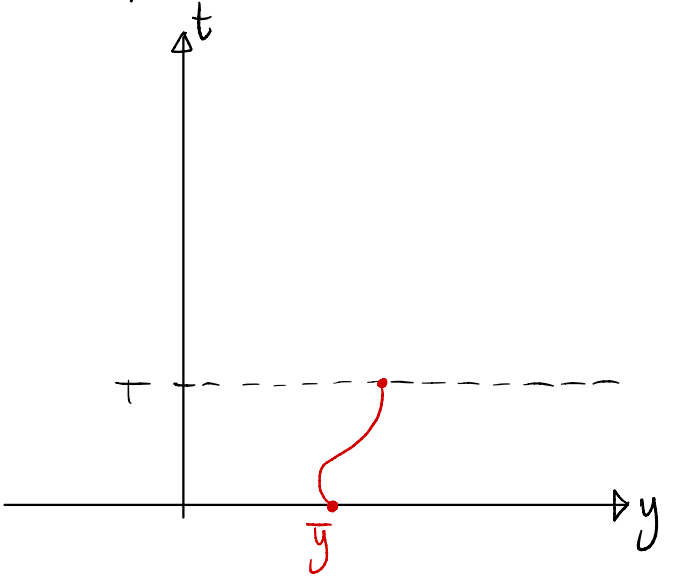
$$F(z)(t) = \overline{y} + \int_0^t f(z(s))\, ds.$$

- Claim: If $T = 1/2K$ then F is a contraction.

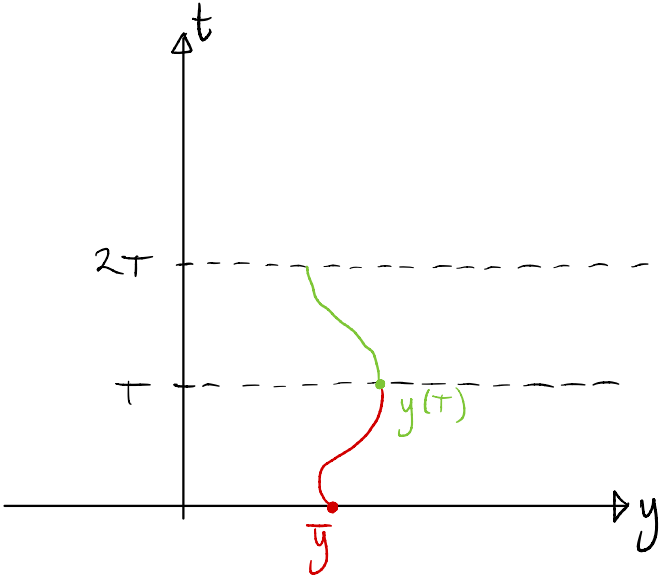
Then $\rho(F(z), F(\tilde{z})) \leq KT\, \rho(z, \tilde{z}) = \frac{1}{2}\, \rho(z, \tilde{z})$

for all $z, \tilde{z} \in C([0, T])$.

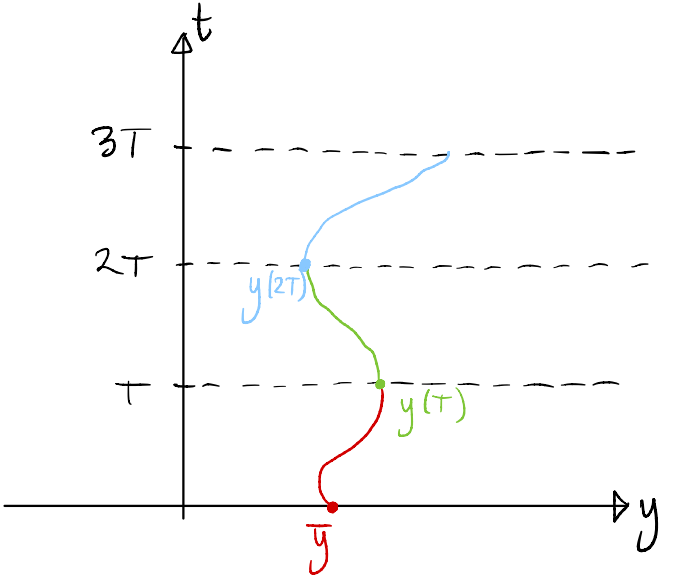
We have shown that F is a contraction of $C([0,T],\mathbb{R})$, which is complete (as $\mathbb{R}$ is complete). Hence, by Banach's fixed point theorem, F has a unique fixed point.



We now repeat the argument in the interval $[T, 2T]$



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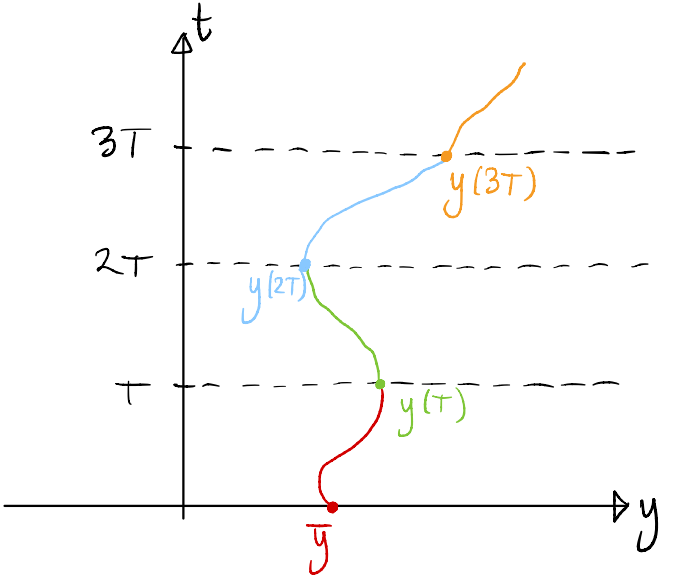


We now repeat the argument in the interval $[T, 2T]$

... and in the interval $[2T, 3T]$

... and so on.

By "gluing" together this string of solutions, we get a solution on all of $[0, \infty)$.



QUESTIONS?

COMMENTS?