# THE ARZELA-ASCOLI THEOREM

Recall:

$$C_b(X,Y) = C(X,Y) \cap B(X,Y)$$
$$= \{\text{bounded, continuous functions } f: X \to Y\}$$

$(C_b(X,Y), \rho)$ is a metric space

**Problem:** What are the compact subsets of $C_b(X,Y)$?

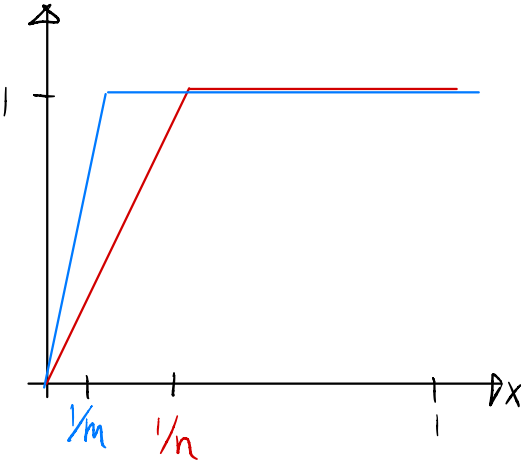
We will answer this when $X$ is compact and $Y = \mathbb{R}^m$.

**Example:** If $f_n(x) = \begin{cases} nx & 0 \le x \le 1/n \\ 1 & 1/n < x, \end{cases}$ for $x \in [0,1]$, $n > 0$

then $\mathcal{F} = \{f_n : n > 0\} \subseteq C([0,1])$ is <u>not</u> compact.

We have $f_n \xrightarrow[n \to \infty]{} f$ pointwise, where

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & x > 0. \end{cases}$$




Hence, if $\{f_{n_k}\}_k$ is any subsequence then also $f_{n_k} \xrightarrow[k \to \infty]{} f$ pointwise.

But $f$ is discontinuous, hence $\{f_{n_k}\}_k$ cannot converge uniformly.

**Def:** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces.

A subset $\mathcal{F} \subseteq C(X, Y)$ is (uniformly) equicontinuous if for every $\varepsilon > 0$ there is some $\delta > 0$ such that

$$d_Y(f(x), f(y)) < \varepsilon$$

for all $x, y \in X$ with $d_X(x, y) < \delta$ and all $f \in \mathcal{F}$.

**Example:**

The set

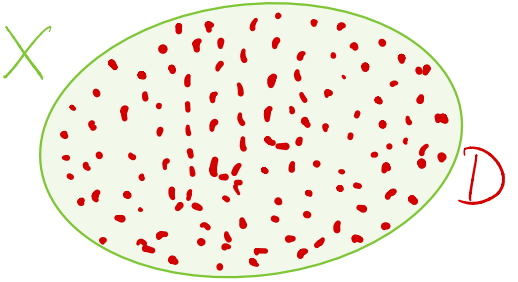
$$\mathcal{F} = \{\text{all } f : X \to Y \text{ satisfying } d_Y(f(x), f(y)) \leq 5\, d_X(x, y)\ \forall x, y \in X\}$$

is equicontinuous.

Recall:

If $(X, d)$ is a metric space then $D \subseteq X$ is <u>dense</u> in $X$ if

$$\bigcup_{x \in D} B(x; \varepsilon) = X \quad \text{for all } \varepsilon > 0.$$




Put a different way: $\forall\ y \in X$ and $\varepsilon > 0$ there is $x \in D$ so that $d(x, y) < \varepsilon$.

Recall:

$(X, d)$ is <u>separable</u> if there exists a dense, countable subset $D \subseteq X$.

<u>Prop.</u>: Every compact set is separable.

**Lemma:** Let $(X, d)$ be compact, let $\mathcal{F} \subseteq C(X, \mathbb{R}^m)$ be equicontinuous, let $D \subseteq X$ be dense, and let $\{g_n\}_n$ be a sequence in $\mathcal{F}$. Then TFAE:

(i) $\{g_n\}_n$ converges uniformly

(ii) $\{g_n(a)\}_n$ converges for all $a \in D$.

**(i) $\Rightarrow$ (ii):** If $g_n \xrightarrow[n\to\infty]{} g$ uniformly then $g_n \xrightarrow[n\to\infty]{} g$ pointwise, so (ii) follows.

**Lemma:** Let $(X, d)$ be compact, let $\mathcal{F} \subseteq C(X, \mathbb{R}^m)$ be equicontinuous, let $D \subseteq X$ be dense, and let $\{g_n\}_n$ be a sequence in $\mathcal{F}$. Then TFAE:

(i) $\{g_n\}_n$ converges uniformly

(ii) $\{g_n(a)\}_n$ converges for all $a \in D$.

(ii) $\Rightarrow$ (i): Let $\varepsilon > 0$ and let $\delta > 0$ be such that

$$\|f(x) - f(y)\| < \varepsilon \quad \forall\, x, y \in X \text{ with } d(x,y) < \delta \text{ and all } f \in \mathcal{F}.$$

Then $\{B(a;\delta) : a \in D\}$ is an open covering of $X$, so there are $a_1, \dots, a_k \in D$ so that $\bigcup_{i=1}^{k} B(a_i;\delta) = X$.

**Lemma:** Let $(X, d)$ be compact, let $\mathcal{F} \subseteq C(X, \mathbb{R}^m)$ be equicontinuous, let $D \subseteq X$ be dense, and let $\{g_n\}_n$ be a sequence in $\mathcal{F}$. Then TFAE:

(i) $\{g_n\}_n$ converges uniformly

(ii) $\{g_n(a)\}_n$ converges for all $a \in D$.

Since $\{g_n(a_i)\}_n$ converges for $i = 1, \dots, k$, there are $N_1, \dots, N_k \in \mathbb{N}$ so that $\|g_n(a_i) - g_m(a_i)\| < \varepsilon \quad \forall\, n, m \geq N_i$, for all $i = 1, \dots, k$.

Let $N = \max(N_1, \dots, N_k)$.

If $x \in X$ then there is an $i \in \{1, \dots, k\}$ so that $x \in B(a_i; \delta)$.

**Lemma:** Let $(X, d)$ be compact, let $\mathcal{F} \subseteq C(X, \mathbb{R}^m)$ be equicontinuous, let $D \subseteq X$ be dense, and let $\{g_n\}_n$ be a sequence in $\mathcal{F}$. Then TFAE:

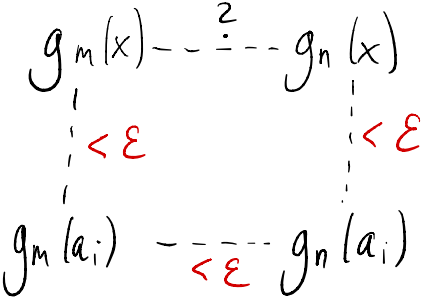
(i) $\{g_n\}_n$ converges uniformly

(ii) $\{g_n(a)\}_n$ converges for all $a \in D$.

If $n, m \geq N$ then

$$\|g_n(x) - g_m(x)\| \leq \|g_n(x) - g_n(a_i)\| + \|g_n(a_i) - g_m(a_i)\| + \|g_m(a_i) - g_m(x)\|$$

$$< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.$$




Hence, $\rho(g_n, g_m) \leq 3\varepsilon$ $\forall$ $n, m \geq N$, so $\{g_n\}_n$ is Cauchy.

**Lemma:** Let $(X, d)$ be compact, let $\mathcal{F} \subseteq C(X, \mathbb{R}^m)$ be equicontinuous, let $D \subseteq X$ be dense, and let $\{g_n\}_n$ be a sequence in $\mathcal{F}$. Then TFAE:

(i) $\{g_n\}_n$ converges uniformly

(ii) $\{g_n(a)\}_n$ converges for all $a \in D$.

- $X$ is compact so $C(X, \mathbb{R}^m) = C_b(X, \mathbb{R}^m)$
- $\mathbb{R}^m$ is complete, so $C_b(X, \mathbb{R}^m)$ is complete.

Hence, $\{g_n\}_n$ is convergent.

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**Arzelà's theorem**: Let $(X,d)$ be compact. Let $\mathcal{F} \subseteq C(X,\mathbb{R}^m)$ be bounded, closed and equicontinuous. Then $\mathcal{F}$ is compact.

**Proof**: Let $\{f_n\}_n$ be a sequence in $\mathcal{F}$. Since $\mathcal{F}$ is bounded, so are $\{f_n\}_n$ and $\{f_n(x)\}_n$ for every $x \in X$.

Let $D = \{a_1, a_2, \dots\} \subseteq X$ be a dense, countable set.

- $\{f_n(a_1)\}_n$ is a bounded sequence in $\mathbb{R}^m$, hence it has a convergent subsequence $\{f_{n_1(k)}(a_1)\}_k$.
- $\{f_{n_1(k)}(a_2)\}_k$ is a bounded sequence in $\mathbb{R}^m$, hence it has a convergent subsequence $\{f_{n_2(k)}(a_2)\}_k$.
  Note that also $\{f_{n_2(k)}(a_1)\}_k$ is convergent.

**Arzelà's theorem:** Let $(X,d)$ be compact. Let $\mathcal{F} \subseteq C(X,\mathbb{R}^m)$ be bounded, closed and equicontinuous. Then $\mathcal{F}$ is compact.

Continue in this fashion to obtain subsequences $\{n_i(k)\}_k$ so that $\{f_{n_1(k)}(a_1)\}_k, \ldots, \{f_{n_i(k)}(a_i)\}_k$ converge.

Let $n(k) = n_k(k)$ $\forall$ $k \in \mathbb{N}$.

Then $\{f_{n(k)}(a_i)\}_k$ converges for <u>all</u> $i \in \mathbb{N}$.

Apply the previous lemma to see that $\{f_{n(k)}\}_k$ converges uniformly.

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## The Arzela-Ascoli theorem:

Let $(X,d)$ be compact and let $\mathcal{F} \subseteq C(X,\mathbb{R}^m)$. Then TFAE:

(i) $\mathcal{F}$ is compact

(ii) $\mathcal{F}$ is closed, bounded and equicontinuous.

(ii) $\Rightarrow$ (i) is Ascoli's theorem.

## The Arzela-Ascoli theorem:

Let $(X,d)$ be compact and let $\mathcal{F} \subseteq C(X,\mathbb{R}^m)$. Then TFAE:

(i) $\mathcal{F}$ is compact

(ii) $\mathcal{F}$ is closed, bounded and equicontinuous.

(i) $\Rightarrow$ (ii): All compact sets are closed and bounded.

Assume that $\mathcal{F}$ is <u>not</u> equicontinuous. Then there is some $\varepsilon > 0$ such that for any $\delta > 0$ there are $f \in \mathcal{F}$ and $x, y \in X$ with $d(x,y) < \delta$ but $\|f(x) - f(y)\| \geq \varepsilon$.

Let $\delta = \frac{1}{n}$ for $n \in \mathbb{N}$ and let $x_n, y_n$ and $f_n$ be as above.

## The Arzela-Ascoli theorem:

Let $(X,d)$ be compact and let $\mathcal{F} \subseteq C(X,\mathbb{R}^m)$. Then TFAE:

(i) $\mathcal{F}$ is compact

(ii) $\mathcal{F}$ is closed, bounded and equicontinuous.

Both $X$ and $\mathcal{F}$ are compact, so there are convergent subsequences $x_{n_k} \xrightarrow[k\to\infty]{} x \in X$ and $f_{n_k} \xrightarrow[k\to\infty]{} f \in \mathcal{F}$.

Since $d(x_n, y_n) < \frac{1}{n}$, also $y_{n_k} \xrightarrow[k\to\infty]{} x$.

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Then

$$\begin{aligned}
\varepsilon &\leq \|f_{n_k}(x_{n_k}) - f_{n_k}(y_{n_k})\| \\
&\leq \|f_{n_k}(x_{n_k}) - f(x_{n_k})\| + \|f(x_{n_k}) - f(y_{n_k})\| + \|f(y_{n_k}) - f_{n_k}(y_{n_k})\| \\
&\leq \rho(f_{n_k}, f) + \|f(x_{n_k}) - f(y_{n_k})\| + \rho(f, f_{n_k}) \\
&\longrightarrow 0 \qquad (k \to \infty)
\end{aligned}$$





QUESTIONS ?

COMMENTS ?