NORMED VECTOR SPACES

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Before you continue, re-watch the background video "6. Vector spaces and norms"

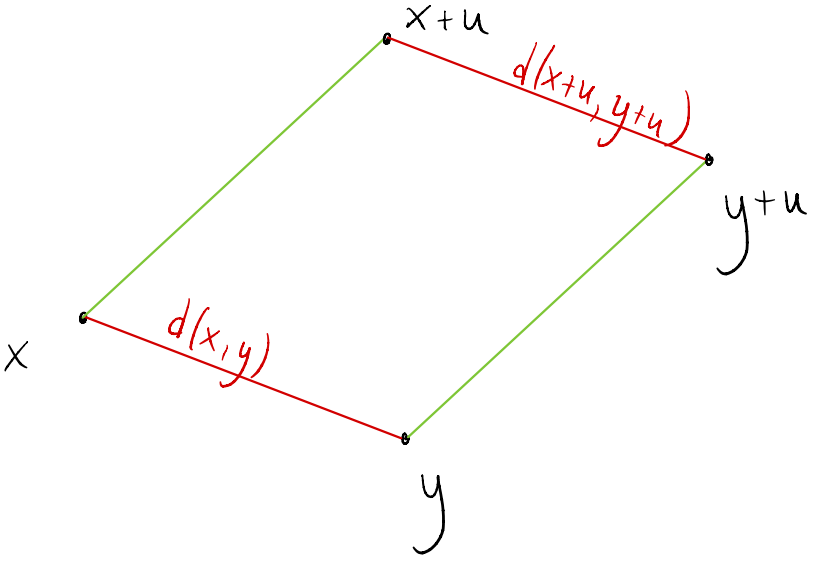
Def: Let V be a vector space

- $x_1, \dots, x_n \in V$ are <u>linearly independent</u> if the only scalars $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that $\alpha_1 x_1 + \dots + \alpha_n x_n = 0$ are $\alpha_1 = \dots = \alpha_n = 0$.
- The <u>span</u> of $\{x_1, \dots, x_n\}$ is
$$\text{span}\{x_1, \dots, x_n\} = \{\alpha_1 x_1 + \dots + \alpha_n x_n : \alpha_1, \dots, \alpha_n \in \mathbb{R}\}$$
- $\{x_1, \dots, x_n\}$ is a <u>basis for V</u> if $x_1, \dots, x_n$ are linearly independent and if $\text{span}\{x_1, \dots, x_n\} = V$.
- V is <u>finite-dimensional</u> if it has a finite basis. If so, $\dim(V)$ denotes the number of elements in any basis for V.

If $(V, \|\cdot\|)$ is a normed vector space then (V, d) is a metric space, where $d(x,y) = \|x-y\|$ (the induced metric).

The induced metric space is "homogeneous" – it looks the same everywhere:

$$d(x,y) = d(u+x, u+y)$$



If $(V, \|\cdot\|)$ is a normed vector space then (V, d) is a metric space, where $d(x,y) = \|x-y\|$ (the induced metric).

Through the induced metric, most statements about metric spaces (open / closed / complete / compact / continuous / convergent / ...) also hold for vector spaces.

Careful: If V is a vector space and $U \subseteq V$ is a subset, then U is not necessarily a vector space!

Prop.:

Let $(V, \|\cdot\|)$ be a vector space, $\{x_n\}_n$ and $\{y_n\}_n$ sequences in V and $\{\alpha_n\}_n$ a sequence in $\mathbb{R}$.

(i) If $x_n \xrightarrow[n\to\infty]{} x$ and $y_n \xrightarrow[n\to\infty]{} y$ then $x_n + y_n \xrightarrow[n\to\infty]{} x+y$

(ii) If $x_n \xrightarrow[n\to\infty]{} x$ and $\alpha_n \xrightarrow[n\to\infty]{} \alpha$ then $\alpha_n x_n \xrightarrow[n\to\infty]{} \alpha x$

(iii) If $x_n \xrightarrow[n\to\infty]{} x$ then $\|x_n\| \xrightarrow[n\to\infty]{} \|x\|$.

Corollary: The function $x \longmapsto \|x\|$ is continuous (as a map from $(V, \|\cdot\|)$ to $(\mathbb{R}, |\cdot|)$).

Def:

A **Banach space** is a complete normed vector space.

Example:

For a sequence $x = (x_1, x_2, x_3, \dots)$ in $\mathbb{R}$ and $p > 0$, write

$$\|x\|_{\ell^p} = \begin{cases} \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} & (p < \infty) \\ \sup \{ |x_1|, |x_2|, \dots \} & (p = \infty). \end{cases}$$

Write $\ell^p(\mathbb{R}) = \{ x : \|x\|_{\ell^p} < \infty \}$.

Then $\left(\ell^p(\mathbb{R}), \|\cdot\|_{\ell^p} \right)$ is a Banach space for every $p \in [1, \infty]$, but <u>not</u> for $p < 1$.

If $(V_1, \|\cdot\|_1), \dots, (V_n, \|\cdot\|_n)$ are normed vector spaces then $(V, \|\cdot\|)$ is a normed vector space, where

$$V = V_1 \times \cdots \times V_n = \{(x_1, \dots, x_n) : x_1 \in V_1, \dots, x_n \in V_n\}$$

$$\|x\| = \|x_1\|_1 + \cdots + \|x_n\|_n .$$

$(V, \|\cdot\|)$ is the Cartesian product of $(V_1, \|\cdot\|_1), \dots, (V_n, \|\cdot\|_n)$

QUESTIONS?
COMMENTS?

Next video: Equivalent norms