NORMED VECTOR SPACES

II

Note: A sequence might converge with respect to one norm, but not another!

Example: Let $x_1 = (1, 0, 0, 0, \dots)$

$x_2 = (1/2, 1/2, 0, 0, \dots)$

$x_3 = (1/3, 1/3, 1/3, 0, 0, \dots)$

$\vdots$

Then x_n lies in both ℓ^1 and ℓ^∞, and

$$\|x_n - 0\|_{\ell^\infty} = \sup_{k \in \mathbb{N}} |x_n(k) - 0| = 1/n \xrightarrow[n \to \infty]{} 0$$

but $$\|x_n - 0\|_{\ell^1} = \sum_{k=1}^{\infty} |x_n(k) - 0| = \sum_{k=1}^{n} 1/n = 1 \not\xrightarrow[n \to \infty]{} 0$$

Def: Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space V are <u>equivalent</u> if there are $c, C > 0$ such that

$$c\|u\|_1 \leq \|u\|_2 \leq C\|u\|_1 \qquad \forall\ u \in V$$

Note: Then also $\tilde{c}\|u\|_2 \leq \|u\|_1 \leq \tilde{C}\|u\|_2 \qquad \forall\ u \in V$

where $\tilde{c} = 1/C$, $\tilde{C} = 1/c$.

Note: If $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, and $\|\cdot\|_2$ and $\|\cdot\|_3$ are equivalent, then so are $\|\cdot\|_1$ and $\|\cdot\|_3$.

Example:
$\|\cdot\|_{\ell^1}$ and $\|\cdot\|_{\ell^\infty}$ are not equivalent on, say, $\ell^1 \cap \ell^\infty$.

Reason: We found $x_1, x_2, \ldots \in \ell^1 \cap \ell^\infty$ such that

$$\|x_n\|_{\ell^\infty} \xrightarrow[n\to\infty]{} 0 \quad \text{but} \quad \|x_n\|_{\ell^1} = 1 \quad \forall n.$$

Proposition:

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be equivalent norms on V. Then:

(i) $\{x_n\}_n$ converges in $\|\cdot\|_1 \iff \{x_n\}_n$ converges in $\|\cdot\|_2$

(ii) a subset U is open in $(V, \|\cdot\|_1) \iff$ it is open in $(V, \|\cdot\|_2)$

(iii) " " closed " " closed

(iv) " " compact " " compact

Theorem: Let V be a finite-dimensional vector space. Then all norms on V are equivalent.

Proof: Let $\{e_1, \dots, e_n\}$ be a basis for V.

Let $T: \mathbb{R}^n \longrightarrow V$ be the linear function

$$T(\alpha) = \sum_{i=1}^{n} \alpha_i e_i .$$

Then T is bijective (as $\{e_1, \dots, e_n\}$ is a basis).

For $x \in V$ define $\|x\| = \|T^{-1}(x)\|_1 = \sum_{i=1}^{n} |\alpha_i|$, where $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ are such that $\sum_{i=1}^{n} \alpha_i e_i = x$.

We will prove that any other norm is equivalent to $\|\cdot\|$.

Theorem: Let V be a finite-dimensional vector space. Then all norms on V are equivalent.

Claim: $\|\cdot\|$ is a norm.

- Clearly, $\|x\| \geq 0 \ \forall x$, and if $\|x\| = 0$ then
$$0 = \|x\| = \|T^{-1}(x)\|_1 = \sum_{i=1}^{n} |\alpha_i| \implies \alpha_1 = \cdots = \alpha_n = 0 \implies x = 0.$$
- T is linear, so T^{-1} is linear, so (if $c \in \mathbb{R}$)
$$\|cx\| = \|T^{-1}(cx)\|_1 = \|c\,T^{-1}(x)\|_1 = |c|\,\|T^{-1}(x)\|_1 = |c| \cdot \|x\|$$
- If $x, y \in V$ then
$$\|x+y\| = \|T^{-1}(x+y)\|_1 = \|T^{-1}(x) + T^{-1}(y)\|_1 \leq \|T^{-1}(x)\|_1 + \|T^{-1}(y)\|_1 = \|x\| + \|y\|$$

Theorem: Let V be a finite-dimensional vector space. Then all norms on V are equivalent.

Let $\|\cdot\|_V$ be any norm on V. If $x \in V$ and $\alpha = T^{-1}(x)$ then

$$\|x\|_V = \|\alpha_1 e_1 + \cdots + \alpha_n e_n\|_V \leq |\alpha_1| \|e_1\|_V + \cdots + |\alpha_n| \|e_n\|_V$$

$$\leq \underbrace{\max_{i=1,\ldots,n} \|e_i\|_V}_{= C} \cdot \underbrace{\left(|\alpha_1| + \cdots + |\alpha_n|\right)}_{= \|\alpha\|_1}$$

$$= C \|x\|$$

Theorem: Let V be a finite-dimensional vector space. Then all norms on V are equivalent.

Conversely, note that $T : \mathbb{R}^n \to V$ is continuous, as

$$\|T(\alpha) - T(\beta)\|_V = \Big\| \sum_{i=1}^n (\alpha_i - \beta_i) e_i \Big\|_V \leq \sum_{i=1}^n |\alpha_i - \beta_i| \, \|e_i\|_V$$

$$\leq \max_{i=1,\dots,n} \|e_i\|_V \cdot \|\alpha - \beta\|_1 = C \|\alpha - \beta\|_1$$

In particular, the function $\alpha \longmapsto \|T(\alpha)\|_V$ is continuous.

Hence, $\alpha \longmapsto \|T(\alpha)\|_V$ attains a minimum on the compact set

$$S = \{\alpha \in \mathbb{R}^n : \|\alpha\|_1 = 1\},$$

say, $\underbrace{\|T(\alpha)\|_V}_{= c} \leq \|T(\alpha)\|_V \quad \forall \ \alpha \in S.$

Theorem: Let V be a finite-dimensional vector space. Then all norms on V are equivalent.

If $x \in V$ and $\alpha = T^{-1}(x)$ then

$$\frac{\|x\|_V}{\|x\|} = \frac{1}{\|\alpha\|_1} \left\| \sum_{i=1}^{n} \alpha_i e_i \right\|_V = \left\| \sum_{i=1}^{n} \frac{\alpha_i}{\|\alpha\|_1} e_i \right\|_V = \left\| T\left(\frac{\alpha}{\|\alpha\|_1}\right) \right\|_V \geq c$$

since $\frac{\alpha}{\|\alpha\|_1} \in S$. Thus:

$$c\|x\| \leq \|x\|_V \leq C\|x\| \qquad \forall\ x \in V$$

so $\|\cdot\|$ and $\|\cdot\|_V$ are equivalent.

∎

QUESTIONS?

COMMENTS?