SERIES AND BASES

Def: Let V be a vector space

- $x_1, \dots, x_n \in V$ are <u>linearly independent</u> if the only scalars $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that $\alpha_1 x_1 + \dots + \alpha_n x_n = 0$ are $\alpha_1 = \dots = \alpha_n = 0$.
- The <u>span</u> of $\{x_1, \dots, x_n\}$ is
$$\text{span}\{x_1, \dots, x_n\} = \{\alpha_1 x_1 + \dots + \alpha_n x_n : \alpha_1, \dots, \alpha_n \in \mathbb{R}\}$$
- $\{x_1, \dots, x_n\}$ is a <u>basis for V</u> if $x_1, \dots, x_n$ are linearly independent and if $\text{span}\{x_1, \dots, x_n\} = V$.
- V is <u>finite-dimensional</u> if it has a finite basis. If so, $\dim(V)$ denotes the number of elements in any basis for V.

Example:

$$\mathbb{P}_n = \{ p : \mathbb{R} \to \mathbb{R} \;:\; p(x) = a_0 + a_1 x + \cdots + a_n x^n \quad \forall\, x \in \mathbb{R} \text{ for } a_0, \ldots, a_n \in \mathbb{R} \}$$

is finite-dimensional, with basis e.g.

- $\{p_0, \ldots, p_n\}$, where $p_k(x) = x^k$ (the monomials)
- $\{p_{n,0}, \ldots, p_{n,n}\}$, where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ (Bernstein polynomials)

Example:

$\mathbb{P} = \bigcup_{n=1}^{\infty} \mathbb{P}_n$ is infinite-dimensional, since $p_0, \ldots, p_n$ are linearly independent for <u>any</u> $n \in \mathbb{N}$.

Definition

If V is a vector space then a subset $E \subseteq V$ is a Hamel basis for V if for every $x \in V$ there are unique elements $e_1, \dots, e_n \in E$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that

$$x = \sum_{k=1}^{n} \alpha_k e_k .$$

However Hamel bases are not easy to do analysis with (for instance, if V is infinite-dimensional then E must be uncountable). We will therefore not work with these here.

Definition:

Let $(V, \|\cdot\|)$ be a normed vector space and let $u_1, u_2, \ldots \in V$.

We say that the series $\sum_{k=1}^{\infty} u_k$ converges if $\{s_n\}_n$ converges, where $s_n = \sum_{k=1}^{n} u_k$ are its partial sums.

Note:

The value of the series (and whether the series converges at all) depends on the choice of the norm!

Definition:

Let $(V, \|\cdot\|)$ be a normed vector space. A Schauder basis (or just basis) is a countable subset $\{e_1, e_2, \dots\}$ of V so that for every $x \in V$, there is a unique sequence $\{\alpha_n\}_n$ in $\mathbb{R}$ such that $\sum_{k=1}^{\infty} \alpha_n e_n = x$.

Note:

A Schauder basis for one norm $\|\cdot\|_1$ might not be a Schauder basis for another norm $\|\cdot\|_2$.

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Exercise:

Every Schauder basis is linearly independent.

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The set $\{e_1, e_2, \dots\}$, where $e_n(k) = \begin{cases} 1 & (k = n) \\ 0 & (k \neq n) \end{cases}$, is a Schauder basis for ℓ^p for all $p \in [1, \infty)$, but <u>not</u> $p = \infty$.

(In fact, it can be shown that ℓ^∞ <u>does not have</u> a Schauder basis. In this sense, ℓ^∞ is <u>very big</u>.)

Thus, objects in ℓ^p $(p < \infty)$ contain "most of their information" in the first few elements, while objects in ℓ^∞ contain information arbitrarily far out into the sequence.

Definition

A series $\sum_{k=1}^{\infty} u_k$ is <u>absolutely convergent</u> if $\sum_{k=1}^{\infty} \|u_k\|$ is convergent.

Proposition:

Let $(V, \|\cdot\|)$ be a normed vector space. Then TFAE:

(i) $(V, \|\cdot\|)$ is complete

(ii) every absolutely convergent series is convergent.

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(ii) $\Rightarrow$ (i): We will skip this.

Proposition:

Let $(V, \|\cdot\|)$ be a normed vector space. Then TFAE:

(i) $(V, \|\cdot\|)$ is complete

(ii) every absolutely convergent series is convergent.

(i) $\Rightarrow$ (ii): Consider an absolutely convergent series $\sum_{k=1}^{\infty} u_k$.

Then $\sum_{k=1}^{\infty} \|u_k\| < \infty$, so $\sum_{k=N}^{\infty} \|u_k\| \xrightarrow[N\to\infty]{} 0$.

Define $s_n = \sum_{k=1}^{n} u_k$. If $n, m \geq N$ and, say, $m < n$ then

$$\|s_n - s_m\| = \left\| \sum_{k=m+1}^{n} u_k \right\| \leq \sum_{k=m+1}^{n} \|u_k\| \leq \sum_{k=N}^{\infty} \|u_k\| \xrightarrow[N\to\infty]{} 0$$

so $\{s_n\}_n$ is Cauchy, and hence convergent. ▨

QUESTIONS ?
COMMENTS ?