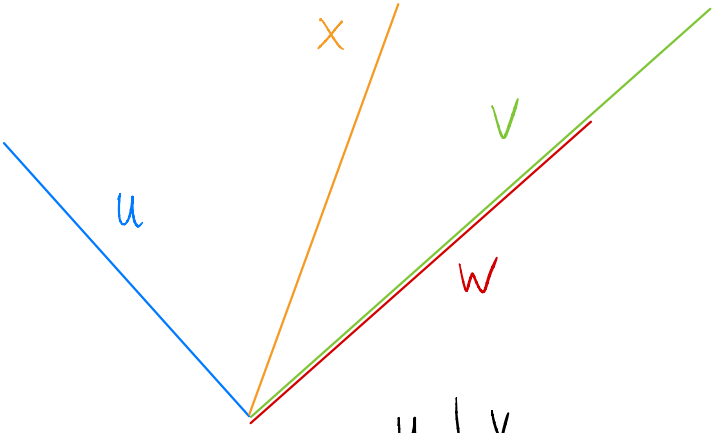
INNER PRODUCT SPACES

In Euclidean space we have an intuitive idea of "parallel", "orthogonal" and "partial overlap".



$u \perp v$

$v \parallel w$

$u \perp w$

$x = \alpha u + \beta v$

To make sense of these in a general vector space V, we need some more structure on V.

Recall:

For $x \in \mathbb{R}^n$, the 2-norm (or canonical norm) can be written

$$\|x\|_2 = \sqrt{x \cdot x} = \sqrt{x_1^2 + \cdots + x_n^2}$$

The dot product $x \cdot y = x_1 y_1 + \cdots + x_n y_n$ can tell us whether x, y are parallel, orthogonal, or neither.

Definition

Let V be a vector space over $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$).

An inner product on V is a function $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{K}$ such that:

(i) $\langle u, v \rangle = \overline{\langle v, u \rangle}$ $\quad \forall\ u, v \in V$

(ii) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ $\quad \forall\ u, v, w \in V$

(iii) $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$ $\quad \forall\ u, v \in V$ and $\alpha \in \mathbb{K}$

(iv) $\langle u, u \rangle \geq 0$ $\quad \forall\ u \in V$, and $\langle u, u \rangle = 0 \Leftrightarrow u = 0$.

We call $(V, \langle \cdot, \cdot \rangle)$ an inner product space.

Then:

(v) $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$ $\quad \forall\ u, v, w \in V$

(vi) $\langle u, \alpha v \rangle = \overline{\alpha} \langle u, v \rangle$ $\quad \forall\ u, v \in V$ and $\alpha \in \mathbb{K}$

(vii) $\langle \alpha u, \alpha v \rangle = |\alpha|^2 \langle u, v \rangle$ $\quad \forall\ u, v \in V$ and $\alpha \in \mathbb{K}$

(viii) by (i), $\langle u, u \rangle = \overline{\langle u, u \rangle}$, so $\langle u, u \rangle \in \mathbb{R}$ for all $u \in V$.

Property (iv) says that this real number is nonnegative.

Examples:

- $V = \mathbb{R}^n$ with $\langle u, v \rangle = u_1 v_1 + \cdots + u_n v_n$
- $V = \mathbb{C}^n$ with $\langle u, v \rangle = u_1 \overline{v_1} + \cdots + u_n \overline{v_n}$
- Let $V = \ell^p(\mathbb{R})$ with $\langle u, v \rangle = \sum_{k=1}^{\infty} u_k v_k$. By Hölder's inequality, $\sum_{k=1}^{\infty} |u_k v_k| \leq \|u\|_{\ell^p} \|v\|_{\ell^q}$, where $\frac{1}{p} + \frac{1}{q} = 1$.
  But we only know $v \in \ell^p$, i.e. $\|v\|_{\ell^p} < \infty$, so we need $p = q \iff p = 2$.

Thus, $\langle u, v \rangle = \sum_{k=1}^{\infty} u_k v_k$ is an inner product on $\ell^2(\mathbb{R})$.

Example:

$V = C([a,b], \mathbb{R})$ with $\langle u, v \rangle = \int_a^b u(t)v(t)\,dt$

(the L^2 inner product)

Definition

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. The norm induced by $\langle \cdot, \cdot \rangle$ is the function

$$\|u\| = \sqrt{\langle u, u \rangle} \qquad \text{for } u \in V.$$

We will soon check that $\|\cdot\|$ is indeed a norm.

Exercise: Check that $\|\cdot\|$ satisfies positivity and homogeneity.

Definition

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space.

(i) $u, v \in V$ are orthogonal $(u \perp v)$ if $\langle u, v \rangle = 0$

(ii) $u, v \in V$ are parallel $(u \parallel v)$ if there is some $\alpha \in \mathbb{K}$ such that $u = \alpha v$ or $v = \alpha u$.

Pythagoras' theorem

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space, let $\|\cdot\|$ be the induced norm. If $u_1, \dots, u_n \in V$ are orthogonal, then

$$\|u_1 + \cdots + u_n\|^2 = \|u_1\|^2 + \cdots + \|u_n\|^2$$

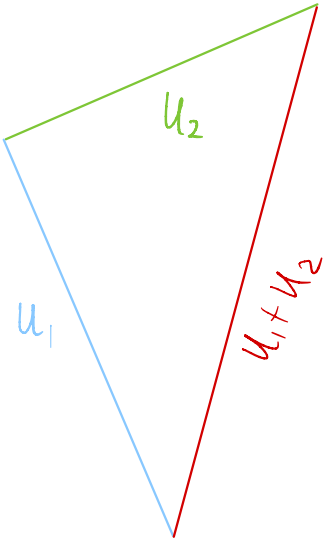
Proof:

$$\left\| \sum_{i=1}^{n} u_i \right\|^2 = \left\langle \sum_{i=1}^{n} u_i \, , \sum_{j=1}^{n} u_j \right\rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle u_i, u_j \rangle = \sum_{i=1}^{n} \langle u_i, u_i \rangle$$

$\langle u_i, u_j \rangle = 0$ if $j \neq i$

$$= \sum_{i=1}^{n} \|u_i\|^2 .$$



Proposition (projection)

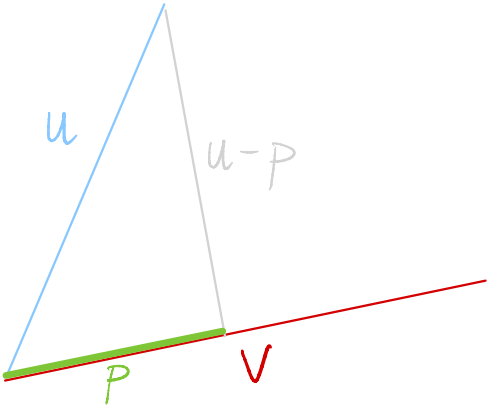
Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space, let $u, v \in V$ be nonzero. Then $p = \frac{\langle u, v \rangle}{\langle v, v \rangle} v$ is the unique element in V satisfying both $p \parallel v$ and $(u - p) \perp v$.

Proof:

If $q \in V$ satisfies both $q \parallel v$ and $(u - q) \perp v$ then $q = \alpha v$ for some $\alpha \in \mathbb{K}$, and

$$0 = \langle u - q, v \rangle = \langle u, v \rangle - \langle \alpha v, v \rangle$$

$$\Longrightarrow \quad \alpha = \frac{\langle u, v \rangle}{\langle v, v \rangle}, \quad \text{so} \quad q = p. \qquad \blacksquare$$



Proposition (Cauchy-Schwarz inequality)

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space, let $\|\cdot\|$ be the induced norm. Then

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\| \qquad \forall\ u, v \in V.$$

Proof: If $u = 0$ or $v = 0$ then the inequality is true. Otherwise, let $p = \frac{\langle u, v \rangle}{\langle v, v \rangle} v$. Then $v \perp (u - p)$, so $p \perp (u - p)$, so

$$\|u\|^2 = \|p + (u-p)\|^2 = \|p\|^2 + \|u-p\|^2 \geq \|p\|^2 = \frac{|\langle u, v \rangle|^2}{|\langle v, v \rangle|^2} \|v\|^2$$

$$= \frac{|\langle u, v \rangle|^2}{\|v\|^2} \quad \Longrightarrow \quad |\langle u, v \rangle|^2 \leq \|u\|^2 \|v\|^2.$$



Theorem

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and let $\|\cdot\|$ be the induced norm. Then $\|\cdot\|$ is a norm.

Proof: We have already checked positivity and homogeneity.

For the triangle inequality, let $u, v \in V$. Then

$$\|u+v\|^2 = \langle u+v, u+v \rangle = \|u\|^2 + \langle u, v \rangle + \langle v, u \rangle + \|v\|^2$$

$$\leq \|u\|^2 + 2\|u\|\,\|v\| + \|v\|^2 = \left(\|u\| + \|v\|\right)^2.$$

Cauchy-Schwarz

∎

We practically always equip an i.p. space $(V, \langle \cdot, \cdot \rangle)$ with its induced norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$.

Thus, all statements about metric spaces and vector spaces also hold for inner product spaces.

Definition

A Hilbert space is an i.p. space $(V, \langle \cdot, \cdot \rangle)$ for which $(V, \|\cdot\|)$ is complete.

Proposition

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Then:

(i) the functions $x \longmapsto \langle x, u \rangle$ and $x \longmapsto \langle u, x \rangle$ are continuous for any $u \in V$

(ii) if $u_n \xrightarrow[n \to \infty]{} u$ then $\langle u_n, v \rangle \longrightarrow \langle u, v \rangle$ $\forall v \in V$.

(iii) if $\sum_{n=1}^{\infty} u_n$ converges then $\left\langle \sum_{n=1}^{\infty} u_n, v \right\rangle = \sum_{n=1}^{\infty} \langle u_n, v \rangle$ $\forall v \in V$.

Proof: Exercise!

QUESTIONS?

COMMENTS?