LINEAR OPERATORS I

Nomenclature

The terms map, mapping, function, functional and operator all refer to functions $f: X \to Y$.

- map / mapping / function are general terms
- functionals are functions $f: V \to \mathbb{R}$ or $f: V \to \mathbb{C}$, where V is a vector space
- operators are functions $f: V \to W$ where V and W are vector spaces.

Definition

Let V, W be vector spaces over either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

A function $A: V \to W$ is a <u>linear operator</u> if

(i) $A(\alpha u) = \alpha A(u) \quad \forall\ \alpha \in \mathbb{K}$ and $u \in V$

(ii) $A(u+v) = A(u) + A(v) \quad \forall\ u, v \in V$.

Thus, a linear operator <u>preserves the linear structure</u> of V.

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If $W = \mathbb{R}$ or $\mathbb{C}$ then we usually say <u>linear functional</u> instead of <u>linear operator</u>

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Exercise

It is enough to check that $A(\alpha u + v) = \alpha A(u) + A(v)$

$\forall\ \alpha \in \mathbb{K}$ and $u, v \in V$

Note:

If $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ and $u_1, \dots, u_n \in V$ then

$$A\left(\sum_{i=1}^{n} \alpha_i u_i\right) = \sum_{i=1}^{n} \alpha_i A(u_i).$$

This is not always true for series ("infinite sums")

Example:

If $V = \mathbb{R}^m$, $W = \mathbb{R}^n$ then $A : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a linear operator iff there is some $M \in \mathbb{R}^{n \times m}$ so that

$$A(u) = Mu \qquad \forall\ u \in \mathbb{R}^m$$

Indeed, if $e_1, \dots, e_m \in \mathbb{R}^m$ is the canonical basis in $\mathbb{R}^m$ then $A(u) = A\left(\sum_{i=1}^m u_i e_i\right) = \sum_{i=1}^m u_i \underbrace{A(e_i)}_{= V_i} = \sum_{i=1}^m u_i v_i = Mu$

where $M = (v_1\ v_2\ \cdots\ v_m)$.

This is not true in infinite-dimensional spaces!

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For this reason we often write "Au" instead of "$A(u)$". This does not mean that "Au" involves any type of multiplication!

Example

$V = C([a,b], \mathbb{R})$, $A : V \to V$ is the map

$$A(u)(t) = \int_a^t u(s)\,ds \qquad \forall\ t \in [a,b].$$

Then $t \mapsto A(u)(t)$ is continuous, so indeed $A(u) \in V$ $\forall\ u \in V$. Next,

$$A(\alpha u + v)(t) = \int_a^t (\alpha u + v)(s)\,ds = \int_a^t \alpha u(s) + v(s)\,ds$$

$$= \alpha \int_a^t u(s)\,ds + \int_a^t v(s)\,ds = \alpha A(u)(t) + A(v)(t) = \big(\alpha A(u) + A(v)\big)(t)$$

$\forall\ t \in [a,b]$, so $A(\alpha u + v) = \alpha A(u) + A(v)$. Hence, A is linear.

Exercise:

Let $V = C^\infty((a,b),\mathbb{R}) = \{f : (a,b) \to \mathbb{R} : \text{all derivatives } f^{(k)} \text{ for } k \in \mathbb{N} \text{ exist}\}$

and let $A : V \to V$ be given by $A(u) = \frac{du}{dt}$. Then

A is linear.

QUESTIONS?

COMMENTS?

Next video: Continuity of linear operators