LINEAR OPERATORS

II

Recall:

Let V, W be normed vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

$A: V \to W$ is a linear operator (or just is linear)

if $A(\alpha u + v) = \alpha Au + Av$ for all $\alpha \in \mathbb{K}$ and $u, v \in V$.

Definition

Let V, W be normed vector spaces and $A: V \to W$ a linear operator. Then A is bounded if there is some $C > 0$ so that

$$\|A(u)\|_W \leq C \|u\|_V \quad \forall\, u \in V.$$

Careful!

"Bounded" here does not mean "the image is bounded", as before. In fact, the image is unbounded: If $u \in V$ is s.t. $A(u) \neq 0$ then

$$\|A(\alpha u)\|_W = \|\alpha A(u)\|_W = |\alpha| \cdot \underbrace{\|A(u)\|_W}_{\neq 0} \xrightarrow{|\alpha| \to \infty} \infty.$$

Definition

Let V, W be normed vector spaces and $A: V \to W$ a linear operator. Then A is bounded if there is some $C > 0$ so that $\|A(u)\|_W \leq C \|u\|_V \quad \forall\, u \in V$.

Definition

If $A: V \to W$ is a bounded linear operator we define its operator norm

$$\|A\|_{\mathcal{L}} = \sup_{\substack{u \in V \\ u \neq 0}} \frac{\|A(u)\|_W}{\|u\|_V}$$

Proposition

Let $A : V \to W$ be a bounded linear operator. Then

$$\|A\|_{\mathcal{L}} = \sup_{u \in V,\, u \neq 0} \frac{\|A(u)\|_W}{\|u\|_V} = \sup_{v \in V,\, \|v\| = 1} \|A(v)\|_W$$

$$= \inf \{ C > 0 : \|A(u)\|_W \leq C \|u\|_V \quad \forall\, u \in V \}.$$

Proof: If $u \neq 0$ then $\dfrac{\|A(u)\|_W}{\|u\|_V} = \left\| \dfrac{1}{\|u\|_V} A(u) \right\|_W = \left\| A\left(\underbrace{\dfrac{1}{\|u\|_V} u}_{\text{norm 1}} \right) \right\|_W$

$$\leq \sup_{v \in V,\, \|v\| = 1} \|A(v)\|_W$$

so $\|A\|_{\mathcal{L}} \leq \sup_{v \in V,\, \|v\| = 1} \|A(v)\|_W$.

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$$= \inf \{ C > 0 : \|A(u)\|_W \leq C \|u\|_V \quad \forall\, u \in V \}.$$

Proof: Conversely, if $v \in V$ has norm $\|v\|_V = 1$ then

$$\|A(v)\|_W = \frac{\|A(v)\|_W}{\|v\|_V} \leq \sup_{u \in V,\, u \neq 0} \frac{\|A(u)\|_W}{\|u\|_V} = \|A\|_{\mathcal{L}}$$

so $\sup_{\|v\| = 1} \|A(v)\| = \|A\|_{\mathcal{L}}$.

Proposition

Let $A : V \to W$ be a bounded linear operator. Then

$$\|A\|_{\mathcal{L}} = \sup_{u \in V,\, u \neq 0} \frac{\|A(u)\|_W}{\|u\|_V} = \sup_{v \in V,\, \|v\| = 1} \|A(v)\|_W$$

$$= \inf \{ C > 0 : \|A(u)\|_W \leq C \|u\|_V \quad \forall\, u \in V \}.$$

Proof: Next, let $\mathcal{C} = \{ C > 0 : \dots \}$. If $u \neq 0$ then

$$\|A(u)\|_W = \frac{\|A(u)\|_W}{\|u\|_V} \cdot \|u\|_V \leq \|A\|_{\mathcal{L}} \|u\|_V,$$

so $\|A\|_{\mathcal{L}} \in \mathcal{C}$, whence $\|A\|_{\mathcal{L}} \geq \inf \mathcal{C}$.

Proposition

Let $A: V \to W$ be a bounded linear operator. Then

$$\|A\|_{\mathcal{L}} = \sup_{u \in V,\, u \neq 0} \frac{\|A(u)\|_W}{\|u\|_V} = \sup_{v \in V,\, \|v\| = 1} \|A(v)\|_W$$

$$= \inf \{ C > 0 : \|A(u)\|_W \leq C \|u\|_V \quad \forall\, u \in V \}.$$

Proof: Last, if $C \in \mathcal{C}$ then

$$\|A(u)\|_W \leq C\|u\|_V \;\; \forall\, u \in V \implies \frac{\|A(u)\|_W}{\|u\|_V} \leq C \quad \forall\, u \neq 0$$

$$\implies \|A\|_{\mathcal{L}} = \sup_{u \neq 0} \frac{\|A(u)\|_W}{\|u\|_V} \leq C. \quad \text{We conclude } \|A\|_{\mathcal{L}} = \inf \mathcal{C}.$$



Problem 5.4.8

If $V = \mathbb{R}^n$ and $A : V \to W$ is linear then it is automatically bounded.

Thus, only linear operators on infinite dimensional spaces can be unbounded.

Example:

Let $V = C([a,b],\mathbb{R})$ with the supremum norm $\|u\|_\infty = \sup_{t\in[a,b]} |u(t)|$.

Let $A: V \to V$ be given by $A(u)(t) = \int_a^t u(s)\,ds$.

Then A is a bounded linear operator.

Indeed, $|A(u)(t)| \le \int_a^t |u(s)|\,ds \le (t-a)\|u\|_\infty \le (b-a)\|u\|_\infty$,

so $\|A(u)\|_\infty \le C\|u\|_\infty$ where $C = b-a$. In particular,

$\|A\|_\alpha \le C$, but if $u \equiv 1$ then $\|A(u)\|_\infty = \sup_t \left|\int_a^t 1\,ds\right| = \sup_t (t-a)$

$= b-a = C\|u\|_\infty$, so $\|A\|_\alpha \ge C$, whence $\|A\|_\alpha = C = b-a$.

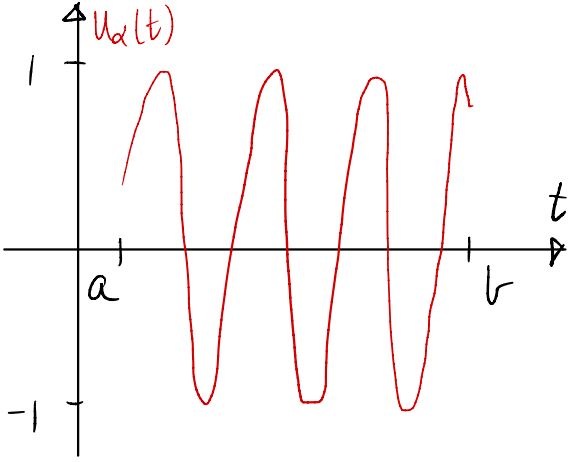
Example:
Let $V = C^\infty([a,b],\mathbb{R})$ with the supremum norm, and let $A: V \to V$, $A(u) = \frac{du}{dt}$. Then A is an unbounded linear op.

Indeed, if $u_\alpha(t) = \sin(\alpha t)$ for $\alpha \in \mathbb{R}$ then $u_\alpha \in V$, and $A(u_\alpha)(t) = \alpha \cos(\alpha t)$. Then $\|u_\alpha\|_\infty = 1$ but $\|A(u_\alpha)\| = |\alpha|$,

so $$\|A\|_\alpha \geq \sup_{\alpha \in \mathbb{R}} \frac{\|A(u_\alpha)\|_\infty}{\|u_\alpha\|_\infty}$$

$$= \sup_{\alpha \in \mathbb{R}} |\alpha| = \infty.$$

Hence, A is unbounded.



Theorem: Let V, W be normed vector spaces and $A : V \to W$ a linear operator. Then TFAE:

(i) A is bounded

(ii) A is Lipschitz continuous

(iii) A is continuous

(iv) A is continuous at 0

We will prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

Proof of (i) $\Rightarrow$ (ii): If $u, v \in V$ then

$$\|A(u) - A(v)\|_W = \|A(u-v)\|_W \leq \|A\|_{\mathcal{L}} \cdot \|u-v\|_V$$

so A is Lipschitz with constant $\|A\|_{\mathcal{L}}$.

Theorem: Let V, W be normed vector spaces and $A : V \to W$ a linear operator. Then TFAE:

(i) A is bounded

(ii) A is Lipschitz continuous

(iii) A is continuous

(iv) A is continuous at 0

Proof of (ii) $\Rightarrow$ (iii): Always true

Proof of (iii) $\Rightarrow$ (iv): Always true

Theorem: Let V, W be normed vector spaces and $A : V \to W$ a linear operator. Then TFAE:

(i) A is bounded

(ii) A is Lipschitz continuous

(iii) A is continuous

(iv) A is continuous at 0

Proof of (iv) $\Rightarrow$ (i): Set $\varepsilon = 1$, and let $\delta > 0$ be s.t. $\|A(u) - A(0)\|_W < \varepsilon$ when $\|u\|_V < \delta$, that is, $\|A(u)\|_W < 1$ when $\|u\|_V < \delta$.

Let $u \in V$ be nonzero. Then

$$\|A(u)\|_W = \frac{2\|u\|_V}{\delta} \left\| A\left(\frac{\delta u}{2\|u\|_V} \right) \right\|_W < \frac{2}{\delta} \|u\|_V,$$

norm $< \delta$, so the whole expression < 1.

so A is bounded with $\|A\|_L < \frac{2}{\delta}$.



Recall: The composition of $A: U \to V$ and $B: V \to W$ is

$B \circ A : U \to W$, $\quad B \circ A(u) = B(A(u))$ for $u \in V$.

For a map $A: U \to U$ we write

$$A^n = \underbrace{A \circ \cdots \circ A}_{n \text{ times}} \qquad (\text{where } n \in \mathbb{N})$$

Just as we write Au to mean $A(u)$, we often write BA to mean $B \circ A$.

The identity map on a set U is $I_U : U \to U$, $I_U(u) = u$.

We often define $A^0 = I_U$ for a map $A : U \to U$.

Exercise: If U is a normed vector space then I_U is a bounded linear operator.

Proposition

Let U, V, W be normed vector spaces over $\mathbb{K}$.

(i) If $A: U \to V$ and $B: V \to W$ are linear then so is $B \circ A$

(ii) If A and B are also bounded then so is $B \circ A$, and $\|B \circ A\|_\alpha \leq \|B\|_\alpha \|A\|_\alpha$

Proof (i) If $\alpha \in \mathbb{K}$ and $u, v \in U$ then

$$B \circ A(\alpha u + v) = B\big(A(\alpha u + v)\big) = B\big(\alpha Au + Av\big) = \alpha B(Au) + B(Av)$$

$$= \alpha\, B \circ A(u) + B \circ A(v).$$

(ii) If $u \in U$ then

$$\|B \circ A(u)\|_W = \|B(A(u))\|_W \leq \|B\|_\alpha \|A(u)\|_V \leq \|B\|_\alpha \|A\|_\alpha \|u\|_U$$



Summary:

- $A: V \to W$ is linear if $A(\alpha u + v) = \alpha A(u) + A(v)$ $\forall \alpha$, $\forall u, v$.
- $A: V \to W$ is a bounded linear operator if it is linear and $\exists\, C > 0$ so that $\|A(u)\|_W \leq C\|u\|_V$ $\forall u$.
- $\|A\|_\alpha \in \mathbb{R}$ is the smallest such constant C (the operator norm of A)
- bounded $\Leftrightarrow$ Lipschitz $\Leftrightarrow$ continuous $\Leftrightarrow$ continuous at 0

QUESTIONS?

COMMENTS?

Next video: The space of all bounded linear operators