LINEAR OPERATORS

III

The space $\mathcal{L}(V,W)$

Recall:

Definition

Let V, W be normed vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

- a function $A : V \to W$ is <u>linear</u> if $A(\alpha u + v) = \alpha A(u) + A(v)$ $\forall \ \alpha \in \mathbb{K}$ and $u, v \in V$.
- a linear function $A : V \to W$ is <u>bounded</u> if the operator norm $\|A\|_{\mathcal{L}} = \sup_{u \neq 0} \frac{\|Au\|_W}{\|u\|_V}$ is finite.
- $A : V \to W$ is bounded $\iff$ it is Lipschitz $\iff$ it is continuous at 0

Definition

For normed vector spaces V, W over $\mathbb{K}$ we let

$$\mathcal{L}(V,W) = \{\text{all bounded linear operators } A : V \to W\}$$

Exercise

$\mathcal{L}(V,W)$ is a vector space over $\mathbb{K}$.

Proposition

$\|\cdot\|_{\mathcal{L}}$ is a norm on $\mathcal{L}(V,W)$.

Proof

- Positivity: It is clear that $\|A\|_{\mathcal{L}} \geq 0 \quad \forall\ A \in \mathcal{L}(V,W)$.

  Moreover,

$$\|A\|_{\mathcal{L}} = 0 \iff \|Au\|_W \leq 0 \cdot \|u\|_V = 0 \quad \forall\ u \in V$$

$$\iff Au = 0 \quad \forall\ u \in V$$

$$\iff A = 0.$$

Proposition

$\|\cdot\|_\alpha$ is a norm on $\mathcal{L}(V,W)$.

Proof

- Homogeneity: If $\alpha \in \mathbb{K}$, $A \in \mathcal{L}(V,W)$ then αA is linear, and

$$\|\alpha A\|_\alpha = \sup_{u \neq 0} \frac{\|(\alpha A)(u)\|_W}{\|u\|_V} = \sup_{u \neq 0} \frac{\|\alpha (Au)\|_W}{\|u\|_V}$$

$$= \sup_{u \neq 0} \frac{|\alpha| \, \|Au\|_W}{\|u\|_V} = |\alpha| \sup_{u \neq 0} \frac{\|Au\|_W}{\|u\|_V} = |\alpha| \cdot \|A\|_\alpha .$$

Proposition

$\|\cdot\|_{\mathcal{L}}$ is a norm on $\mathcal{L}(V,W)$.

Proof

- **Triangle inequality**: If $A, B \in \mathcal{L}(V,W)$ and $u \neq 0$ then

$$\frac{\|(A+B)(u)\|_W}{\|u\|_V} = \frac{\|Au+Bu\|_W}{\|u\|_V} \leq \frac{\|Au\|_W}{\|u\|_V} + \frac{\|Bu\|_W}{\|u\|_V} \leq \|A\|_{\mathcal{L}} + \|B\|_{\mathcal{L}}.$$

Taking supremum over all $u \neq 0$ gives $\|A+B\|_{\mathcal{L}} \leq \|A\|_{\mathcal{L}} + \|B\|_{\mathcal{L}}$.



Proposition
If W is complete then so is $\mathcal{L}(V,W)$.

Proof Let $\{A_n\}_n$ be a Cauchy sequence in $\mathcal{L}(V,W)$, so that for all $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$\|A_n - A_m\| = \sup_{u \neq 0} \frac{\|A_n u - A_m u\|_W}{\|u\|_V} < \varepsilon \qquad \forall \; n, m \geq N.$$

Then $\|A_n u - A_m u\|_W < \varepsilon \|u\|_V$ for all $n, m \geq N$ for every fixed $u \in V$, so $\{A_n u\}_n$ is a Cauchy sequence in W. But W is complete, so $A_n u \xrightarrow[n \to \infty]{} Au$ for some element "Au" $\in W$.

This now defines a function $A : u \longmapsto Au$ from V to W.

Proposition
If W is complete then so is $\mathcal{L}(V,W)$.

Claim: A is a bounded linear operator and $A_n \xrightarrow[n\to\infty]{} A$.

A is linear, since $A(\alpha u+v) = \lim_{n\to\infty} A_n(\alpha u+v)$

$$= \lim_{n\to\infty}\left(\alpha A_n u + A_n v\right) = \alpha \lim_{n\to\infty} A_n u + \lim_{n\to\infty} A_n v = \alpha Au + Av.$$

Next, $\varepsilon\|u\|_V > \|A_n u - A_m u\|_W \xrightarrow[m\to\infty]{} \|A_n u - Au\|_W$

for every $u \in V$, so

$$\|A_n - A\|_{\mathcal{L}} = \sup_{u\neq 0} \frac{\|A_n u - Au\|_W}{\|u\|_V} \leq \varepsilon \qquad \forall\ n \geq N.$$

Thus, $\|A_n - A\|_{\mathcal{L}} \xrightarrow[n\to\infty]{} 0$.

Finally, A is bounded, since $\|A\|_{\mathcal{L}} \leq \|A - A_N\|_{\mathcal{L}} + \|A_N\|_{\mathcal{L}} < \infty$. ▨

QUESTIONS?

COMMENTS?