INVERTIBLE LINEAR OPERATORS

I

Recall

If a function $A: U \to V$ is bijective then there is a unique function $A^{-1}: V \to U$ satisfying $A^{-1}(A(u)) = u$ $\forall\ u \in U$ and $A(A^{-1}(v)) = v$ $\forall\ v \in V$.

We can write these as

$$A^{-1} \circ A = I_V \quad \text{and} \quad A \circ A^{-1} = I_U.$$

For the rest of this video, let U, V, W be normed vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Proposition

If $A : U \to V$ is linear and bijective then A^{-1} is linear.

Proof: Let $x, y \in V$ and $\alpha \in \mathbb{K}$.

Let $u = A^{-1}(x)$ and $v = A^{-1}(y)$. Then

$$x + y = Au + Av = A(u + v)$$

$$\Rightarrow A^{-1}(x+y) = A^{-1}(A(u+v)) = u + v = A^{-1}(x) + A^{-1}(y).$$

If $\alpha \in \mathbb{K}$ then $\alpha x = \alpha Au = A(\alpha u)$

$$\Rightarrow A^{-1}(\alpha x) = A^{-1}(A(\alpha u)) = \alpha u = \alpha A^{-1}(x).$$



Example: The inverse might not be bounded!

Let $U = C^\infty([0,1])$, $V = \{v \in C^\infty([0,1]) : v(0) = 0\}$, both equipped with the supremum norm. Let $A: U \to V$, $A(u)(t) = \int_0^t u(s)\,ds$. Then A is bounded, linear and bijective, but A^{-1} is not bounded.

Example: The inverse might not be bounded!

Let $U = C^\infty([0,1])$, $V = \{v \in C^\infty([0,1]) : v(0) = 0\}$, both equipped with the supremum norm. Let $A: U \to V$, $A(u)(t) = \int_0^t u(s)\,ds$. Then A is bounded, linear and bijective, but A^{-1} is not bounded.

- It is clear that $A(u) \in V \quad \forall\, u \in U$: If u is C^∞ then so is its antiderivative, and $A(u)(0) = \int_0^0 u(s)\,ds = 0$.
- A is linear and bounded, as we have shown.

Example: The inverse might not be bounded!

Let $U = C^\infty([0,1])$, $V = \{v \in C^\infty([0,1]) : v(0) = 0\}$, both equipped with the supremum norm. Let $A: U \to V$, $A(u)(t) = \int_0^t u(s)\,ds$. Then A is bounded, linear and bijective, but A^{-1} is not bounded.

- Claim: $A^{-1} = d/dt$. Indeed, if $u \in U$, $v \in V$ then

$$A\left(\tfrac{d}{dt}(v)\right)(t) = \int_0^t \frac{dv}{dt}(s)\,ds = v(t) - v(0) = v(t),$$

$$\tfrac{d}{dt}\left(A(u)\right)(t) = \frac{d}{dt}\int_0^t u(s)\,ds = u(t)$$

so $A \circ \frac{d}{dt} = I_V$ and $\frac{d}{dt} \circ A = I_U$.

- But $\frac{d}{dt}: V \to U$ is not bounded!

Definition

A function $A \in \mathcal{L}(U, V)$ is <u>invertible</u> if it is bijective and A^{-1} is bounded.

Example

Let $U = \mathbb{R}^m$, $V = \mathbb{R}^n$ and $A : U \to V$ linear.

- Then A is automatically bounded.
- A is linear <u>iff</u> $\exists\ M \in \mathbb{R}^{n \times m}$ s.t. that $A(u) = Mu \quad \forall\ u \in U$.
- A can be bijective <u>only</u> when $m = n$
- If A is bijective then A^{-1} is linear, and hence automatically bounded.
- Necessarily, M is invertible, and $A^{-1}(v) = M^{-1}v \quad \forall\ v \in V$.

Exercise

If $A \in \mathcal{L}(U,V)$ and $B \in \mathcal{L}(V,W)$ are invertible then so is BA, and $(BA)^{-1} = A^{-1}B^{-1}$.

Recall: $BA = B \circ A$, and $\|BA\|_{\mathcal{L}} \leq \|B\|_{\mathcal{L}} \|A\|_{\mathcal{L}}$.

Bounded inverse theorem (thm 5.7.5)

Let U, V be Banach spaces and let $A \in \mathcal{L}(U, V)$ be bijective. Then A^{-1} is bounded.

The proof requires the Baire Category Theorem and will not be taught in this course.

QUESTIONS?
COMMENTS?

Next video: Neumann series