INVERTIBLE LINEAR OPERATORS

II

NEUMANN SERIES

We will study a special case where we can find a formula for the inverse of an operator.

Recall: $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \forall\ x \in \mathbb{R}$ with $|x| < 1$.

Analogously, we will show that

$$(I_U - A)^{-1} = \sum_{k=0}^{\infty} A^k \quad \forall\ A \in \mathcal{L}(U,U) \text{ with } \|A\|_{\mathcal{L}} < 1.$$

This is not that useful – we want to find A^{-1}, not $(I_U - A)^{-1}$.

If $c \in \mathbb{R}$, $c \neq 0$ and $|x-c| < |c|$ then $\left|\frac{c-x}{c}\right| < 1$, so

$$\frac{1}{x} = \frac{1}{c-(c-x)} = \frac{1}{c}\,\frac{1}{1-\frac{c-x}{c}} = \frac{1}{c}\sum_{k=0}^{\infty}\left(\frac{c-x}{c}\right)^k = \frac{1}{c}\sum_{k=0}^{\infty}\left(1-\frac{x}{c}\right)^k$$

so we would hope that if $C \in \mathcal{L}(U,U)$ is invertible and $\|A-C\|_{\mathcal{L}} < \|C\|_{\mathcal{L}}$ then A is invertible, with

$$A^{-1} = C^{-1}\sum_{k=0}^{\infty}\left(I_U - C^{-1}A\right)^k.$$

This is <u>almost</u> true!

Theorem

Let U be a Banach space, let $A \in \mathcal{L}(U,U)$ satisfy $\|A\|_{\mathcal{L}} < 1$.

Then $I_U - A$ is invertible and $(I_U - A)^{-1} = \sum_{k=0}^{\infty} A^k$.

Moreover, $\|(I_U - A)^{-1}\| \leq \frac{1}{1 - \|A\|_{\mathcal{L}}}$.

Proof: Recall that $\mathcal{L}(U)$ is complete. Define $S_n = \sum_{k=0}^{n} A^k \in \mathcal{L}(U,U)$.

If $n, m \geq N$ and, say, $n \geqslant m$ then

$$\|S_n - S_m\|_{\mathcal{L}} = \left\| \sum_{k=m+1}^{n} A^k \right\|_{\mathcal{L}} \leq \sum_{k=m+1}^{n} \|A\|_{\mathcal{L}}^k \leqslant \sum_{k=N}^{\infty} \|A\|_{\mathcal{L}}^k \xrightarrow[N \to \infty]{} 0.$$

Hence, $\{S_n\}_{n \in \mathbb{N}}$ is Cauchy, so it converges, hence $\sum_{k=0}^{\infty} A^k$ converges.

Theorem

Let U be a Banach space, let $A \in \mathcal{L}(U,U)$ satisfy $\|A\|_{\mathcal{L}} < 1$.

Then $I_U - A$ is invertible and $(I_U - A)^{-1} = \sum_{k=0}^{\infty} A^k$.

Moreover, $\|(I_U - A)^{-1}\| \leq \frac{1}{1 - \|A\|_{\mathcal{L}}}$.

We have $(I_U - A) S_n = \sum_{k=0}^{n} A^k - \sum_{k=1}^{n+1} A^k = I_U - A^{n+1}$,

and $\|A^{n+1}\|_{\mathcal{L}} \leq \|A\|_{\mathcal{L}}^{n+1} \xrightarrow[n \to \infty]{} 0$, since $\|A\|_{\mathcal{L}} < 1$. Hence,

$$(I_U - A) \cdot \left(\sum_{k=0}^{\infty} A^k\right) = (I_U - A) \cdot \left(\lim_{n \to \infty} S_n\right) = \lim_{n \to \infty} (I_U - A^{n+1}) = I_U.$$

Likewise, $\left(\sum_{k=0}^{\infty} A^k\right)(I_U - A) = I_U$. This shows that $I_U - A$ is invertible with inverse $\sum_{k=0}^{\infty} A^k$.

Theorem

Let U be a Banach space, let $A \in \mathcal{L}(U,U)$ satisfy $\|A\|_{\mathcal{L}} < 1$.

Then $I_U - A$ is invertible and $(I_U - A)^{-1} = \sum_{k=0}^{\infty} A^k$.

Moreover, $\|(I_U - A)^{-1}\|_{\mathcal{L}} \leq \frac{1}{1 - \|A\|_{\mathcal{L}}}$.

Finally,

$$\|(I_U - A)^{-1}\|_{\mathcal{L}} = \Big\| \lim_{n \to \infty} S_n \Big\|_{\mathcal{L}} = \lim_{n \to \infty} \|S_n\|_{\mathcal{L}} = \lim_{n \to \infty} \Big\| \sum_{k=0}^{n} A^k \Big\|_{\mathcal{L}}$$

$$\leq \lim_{n \to \infty} \sum_{k=0}^{n} \|A\|_{\mathcal{L}}^k \leq \lim_{n \to \infty} \sum_{k=0}^{\infty} \|A\|_{\mathcal{L}}^k = \frac{1}{1 - \|A\|_{\mathcal{L}}}.$$



Corollary

Let U be a Banach space and let $B \in \mathcal{L}(U)$ satisfy $\|I_U - B\| < 1$. Then B is invertible, $B^{-1} = \sum_{k=0}^{\infty} (I_U - B)^k$ and $\|B^{-1}\|_{\mathcal{L}} \leq \frac{1}{1 - \|I_U - B\|_{\mathcal{L}}}$.

Proof: Set $A = I_U - B$ in the previous theorem. ∎

Corollary (Banach's lemma)

Let U be a Banach space, let $C \in \mathcal{L}(U)$ be invertible and set $r = \dfrac{1}{\|C^{-1}\|_{\mathcal{L}}}$. Then every $D \in B_{\mathcal{L}}(C; r)$ is invertible with $\|D^{-1}\|_{\mathcal{L}} \leq \dfrac{1}{r - \|C-D\|_{\mathcal{L}}}$.

Proof: Let $D \in B_{\mathcal{L}}(C; r)$, that is, $\|C-D\|_{\mathcal{L}} < 1/\|C^{-1}\|_{\mathcal{L}}$.

Write $E = C^{-1}D$, and note that $C - D = C(I_U - E)$.

We claim E is invertible. Indeed, $E \in \mathcal{L}(U)$, and

$$\|I_U - E\|_{\mathcal{L}} = \|I_U - C^{-1}D\|_{\mathcal{L}} = \|C^{-1}(C-D)\|_{\mathcal{L}}$$
$$\leq \|C^{-1}\|_{\mathcal{L}} \|C-D\|_{\mathcal{L}} < \frac{1}{r} \cdot r = 1$$

so E is invertible, by the previous corollary.

Corollary (Banach's lemma)

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Moreover, $\|E^{-1}\|_{\mathcal{L}} \leq \frac{1}{1 - \|I_U - E\|_{\mathcal{L}}}$. Then $D = CE$ is also invertible with $D^{-1} = E^{-1}C^{-1}$, and

$$\|D^{-1}\|_{\mathcal{L}} \leq \|E^{-1}\|_{\mathcal{L}} \|C^{-1}\|_{\mathcal{L}} \leq \frac{\|C^{-1}\|_{\mathcal{L}}}{1 - \|I_U - E\|_{\mathcal{L}}} \leq \frac{\|C^{-1}\|_{\mathcal{L}}}{1 - \|C^{-1}\|_{\mathcal{L}} \|C-D\|_{\mathcal{L}}}$$

$$= \frac{1}{r - \|C-D\|_{\mathcal{L}}}.$$



QUESTIONS?

COMMENTS?