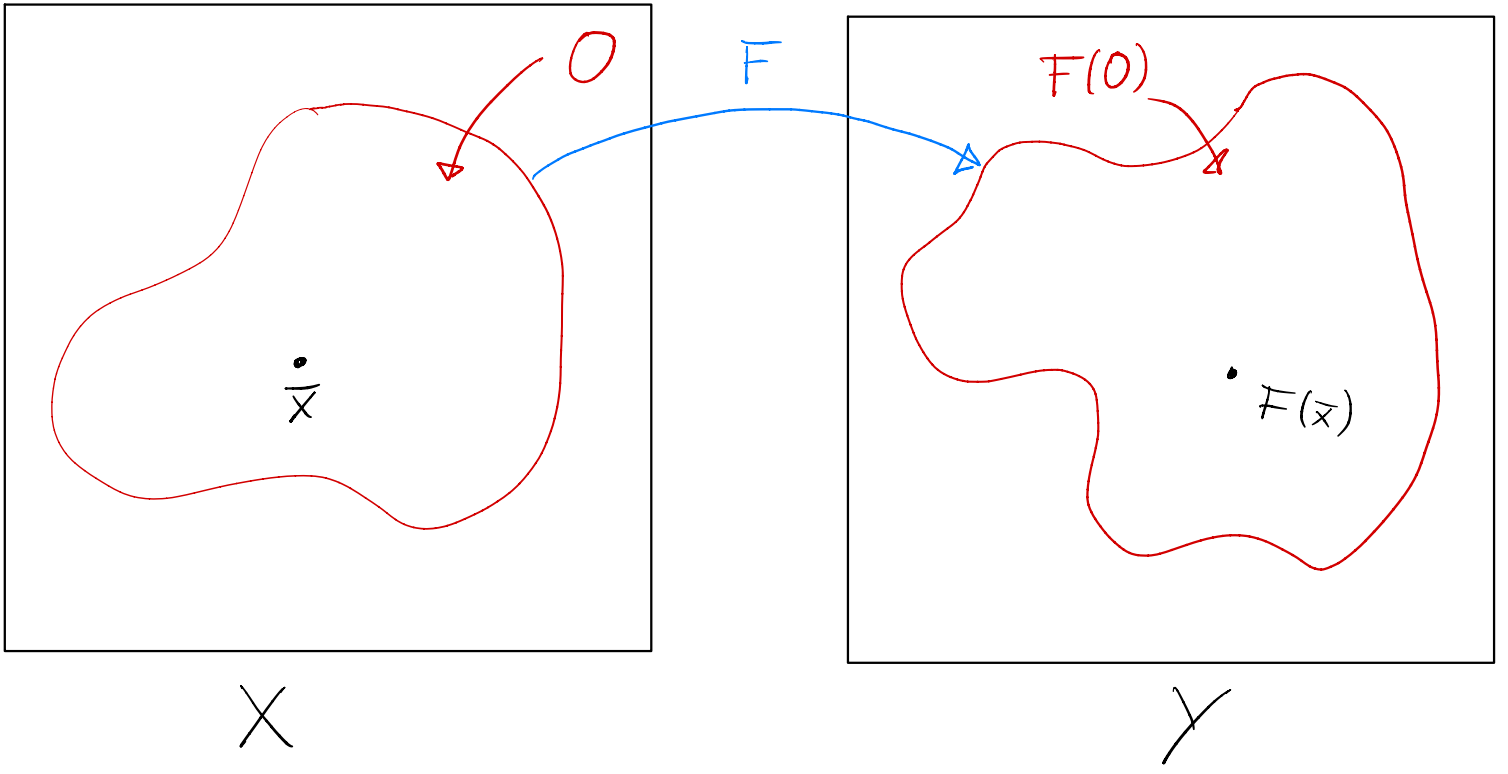
THE FRÉCHET DERIVATIVE I



What is the derivative of F at $\bar{x}$?

Definition

Let X, Y be normed vector spaces, let $O \subseteq X$ be open and let $F: O \to Y$ be a function.

Then F is <u>(Fréchet) differentiable</u> at a point $\bar{x} \in O$ if there is some $A \in \mathcal{L}(X, Y)$ such that

$$\lim_{\substack{h \in X \\ h \to 0}} \frac{\| F(\bar{x}+h) - F(\bar{x}) - A(h) \|_Y}{\|h\|_X} = 0$$

A is the <u>(Fréchet) derivative</u> of F at $\bar{x}$.

Lemma:
If $F: O \to Y$ is differentiable at $\bar{x} \in O$, then the derivative is unique.

Proof
Let $A, B \in \mathcal{L}(X, Y)$ be derivatives of F at $\bar{x}$. Define

$$f(h) := \frac{\left\| \left(F(\bar{x}+h) - F(\bar{x}) - A(h) \right) - \left(F(\bar{x}+h) - F(\bar{x}) - B(h) \right) \right\|_Y}{\|h\|_X}$$

$$= \frac{\|B(h) - A(h)\|_Y}{\|h\|_X} = \left\| (B-A)\left(\frac{h}{\|h\|_X} \right) \right\|_Y .$$

Lemma: If $F: O \to Y$ is differentiable at $\bar{x} \in O$, then the derivative is unique.

On the other hand,

$$f(h) = \frac{\left\| \left(F(\bar{x}+h) - F(\bar{x}) - A(h) \right) - \left(F(\bar{x}+h) - F(\bar{x}) - B(h) \right) \right\|_Y}{\|h\|_X}$$

$$\leq \frac{\|F(\bar{x}+h) - F(\bar{x}) - A(h)\|_Y}{\|h\|_X} + \frac{\|F(\bar{x}+h) - F(\bar{x}) - B(h)\|_Y}{\|h\|_X}$$

$$\xrightarrow[h \to 0]{} 0$$

Lemma: If $F: O \to Y$ is differentiable at $\bar{x} \in O$, then the derivative is unique.

Thus, for any $x \in X$ and $\alpha > 0$, we get

$$\|(B-A)(x)\|_Y = \|x\|_X \left\|(B-A)\left(\frac{\alpha x}{\|\alpha x\|_X}\right)\right\|_Y$$

$$= \|x\|_X \, f(\alpha x) \xrightarrow[\alpha \to 0]{} 0$$

so $(B-A)(x) = 0$. Since x was arbitrary, we get $A = B$.

∎

We will write "$F'(\bar{x})$" instead of "A".

Note

$F'(\bar{x})$ is a bounded linear operator, and

$$\|F'(\bar{x})(h)\|_Y \leq \|F'(\bar{x})\|_{\mathcal{L}} \|h\|_X$$

Example

Let $X = Y = \mathbb{R}$. Then every $A \in \mathcal{L}(X, Y)$ is of the form $A(h) = \alpha h$ for some $\alpha \in \mathbb{R}$. Then

$$\frac{\| F(\bar{x}+h) - F(\bar{x}) - F'(\bar{x})(h) \|_Y}{\|h\|_X} = \frac{|F(\bar{x}+h) - F(\bar{x}) - \alpha h|}{|h|}$$

$$= \left| \frac{F(\bar{x}+h) - F(\bar{x})}{h} - \alpha \right| \xrightarrow[h \to 0]{!} 0$$

$$\Longrightarrow \quad \alpha = \frac{dF}{dx}(\bar{x}), \qquad \text{so} \qquad F'(\bar{x})(h) = \frac{dF}{dx}(\bar{x})\, h$$

Example

Let $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$. If $F: X \to Y$ is differentiable at $\bar{x} \in X$ then, by Taylor's theorem,

$$\frac{\|F(\bar{x}+h) - F(\bar{x}) - \nabla F(\bar{x})\, h\|_2}{\|h\|_2} \xrightarrow[h \to 0]{} 0.$$

Hence, we must have $F'(\bar{x})(h) = \nabla F(\bar{x})\, h$, which is a bounded linear operator.

QUESTIONS?

COMMENTS?

Next video: Properties of Fréchet derivatives