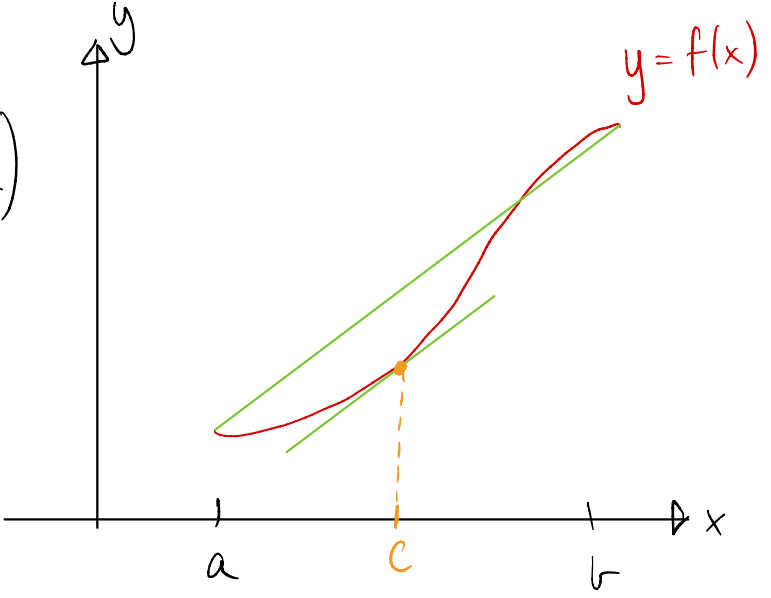
THE MEAN VALUE THEOREM

Recall: If $f: [a,b] \to \mathbb{R}$ is differentiable then there is some $c \in (a,b)$ where

$$f(b) - f(a) = f'(c)(b-a)$$

There is no direct analogue for $F: X \to Y$ (general normed) vector spaces, but often it is sufficient with a "mean value inequality".



Lemma: Let Y be a normed vector space and let $F:[0,1]\to Y$ and $g:[0,1]\to\mathbb{R}$ be continuous in $[0,1]$ and differentiable in $(0,1)$, and satisfy

$$\|F'(t)\|_{\mathcal{L}} \le g'(t) \qquad \forall\ t\in(0,1).$$

Then $$\|F(1)-F(0)\|_Y \le g(1)-g(0).$$

For example, if $\|F'(t)\|_{\mathcal{L}} \le k$ for all t for some $k>0$, let $g(t)=kt$ to get $$\|F(1)-F(0)\|_Y \le k.$$

Lemma: Let Y be a normed vector space and let $F:[0,1]\to Y$ and $g:[0,1]\to\mathbb{R}$ be continuous in $[0,1]$ and differentiable in $(0,1)$, and satisfy

$$\|F'(t)\|_Y \le g'(t) \qquad \forall\ t\in(0,1).$$

Then $$\|F(1)-F(0)\|_Y \le g(1)-g(0).$$

Proof: We aim to prove that $\|F(t)-F(0)\|_Y \le g(t)-g(0)+\varepsilon t$ (*) for every $t\in[0,1]$ and $\varepsilon>0$.

Assume conversely that (*) is not true for some $\varepsilon>0$. Both sides of (*) are continuous, so there is some $t_0\in[0,1)$ where (*) holds for $t=t_0$, but for every $n\in\mathbb{N}$ there is some $t_n\in\left(t_0, t_0+\frac{1}{n}\right)$ where (*) is not true.

Note that $t_n \xrightarrow[n\to\infty]{} t_0$.

Lemma: Let Y be a normed vector space and let $F:[0,1]\to Y$ and $g:[0,1]\to\mathbb{R}$ be continuous in $[0,1]$ and differentiable in $(0,1)$, and satisfy

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Then $$\|F(1)-F(0)\|_Y \le g(1)-g(0).$$

Then
$$\begin{cases} \|F(t_0)-F(0)\|_Y \le g(t_0)-g(0)+\varepsilon t_0 \\ \|F(t_n)-F(0)\|_Y > g(t_n)-g(0)+\varepsilon t_n . \end{cases}$$

Subtract the two ineq:

$$g(t_n)-g(t_0)+\varepsilon(t_n-t_0) < \|F(t_n)-F(0)\|_Y - \|F(t_0)-F(0)\|_Y$$
$$\le \|F(t_n)-F(t_0)\|_Y \qquad \text{(inverse triangle ineq.)}$$

Lemma: Let Y be a normed vector space and let $F:[0,1]\to Y$ and $g:[0,1]\to\mathbb{R}$ be continuous in $[0,1]$ and differentiable in $(0,1)$, and satisfy

$$\|F'(t)\|_Y \le g'(t) \qquad \forall\ t\in(0,1).$$

Then $$\|F(1)-F(0)\|_Y \le g(1)-g(0).$$

$$g(t_n)-g(t_0)+\varepsilon(t_n-t_0) \le \|F(t_n)-F(t_0)\|_Y$$

Divide by t_n-t_0:

$$\frac{g(t_n)-g(t_0)}{t_n-t_0}+\varepsilon \le \frac{\|F(t_n)-F(t_0)-F'(t_0)(t_n-t_0)\|}{t_n-t_0}+\underbrace{\frac{\|F'(t_0)(t_n-t_0)\|}{t_n-t_0}}_{\le \|F'(t_0)\|_Y}$$

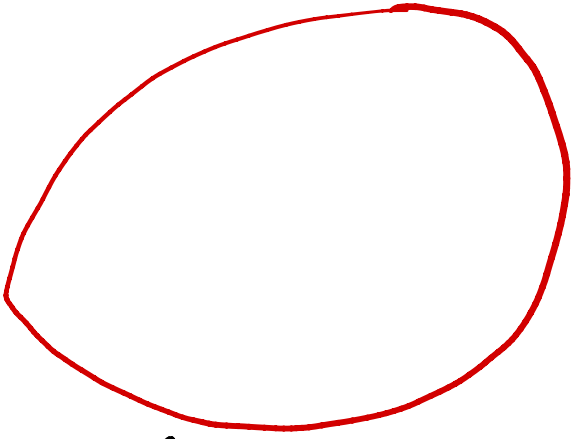
Let $n\to\infty$:

$$g'(t_0)+\varepsilon \le 0+\|F'(t_0)\|_Y \le g'(t_0)$$ ↯ ∎

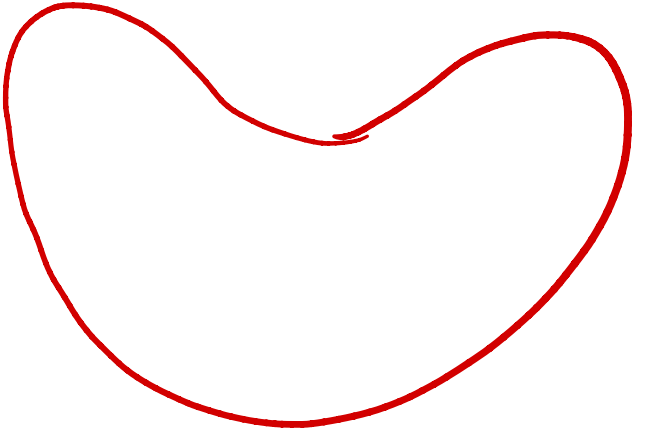
Definition

A subset $C \subseteq X$ of a vector space X is convex if

$$a, b \in C \implies a + t(b-a) \in C \quad \forall\, t \in [0,1].$$



Convex

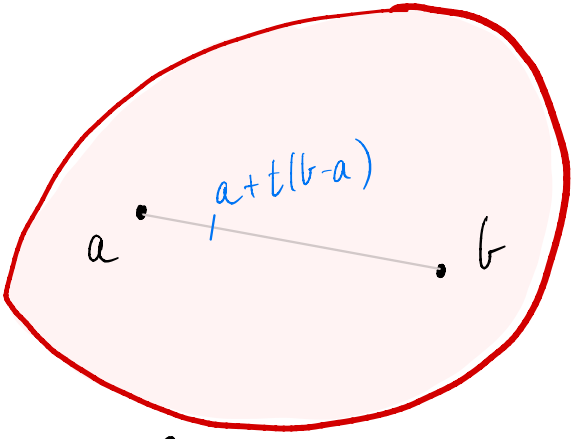


Not convex

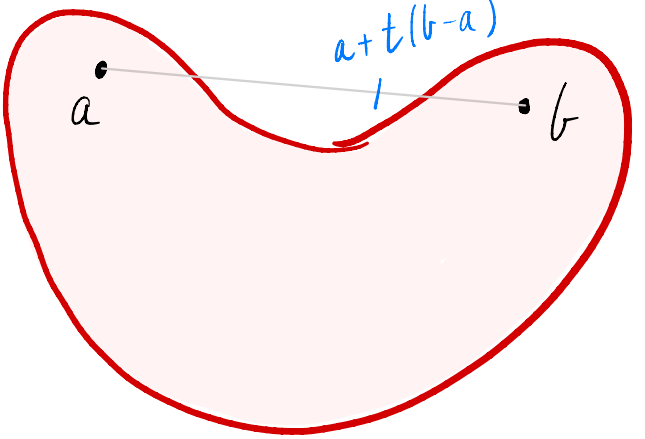
Definition

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Convex



Not convex

Theorem (mean value inequality)

Let X, Y be normed vector spaces, let $C \subseteq X$ be convex, let $\bar{x} \in C$ and let $F: C \to Y$ be differentiable and satisfy $\|F'(x)\|_{\mathcal{L}} \leq k \quad \forall\, x \in C$ for some $k > 0$. Then

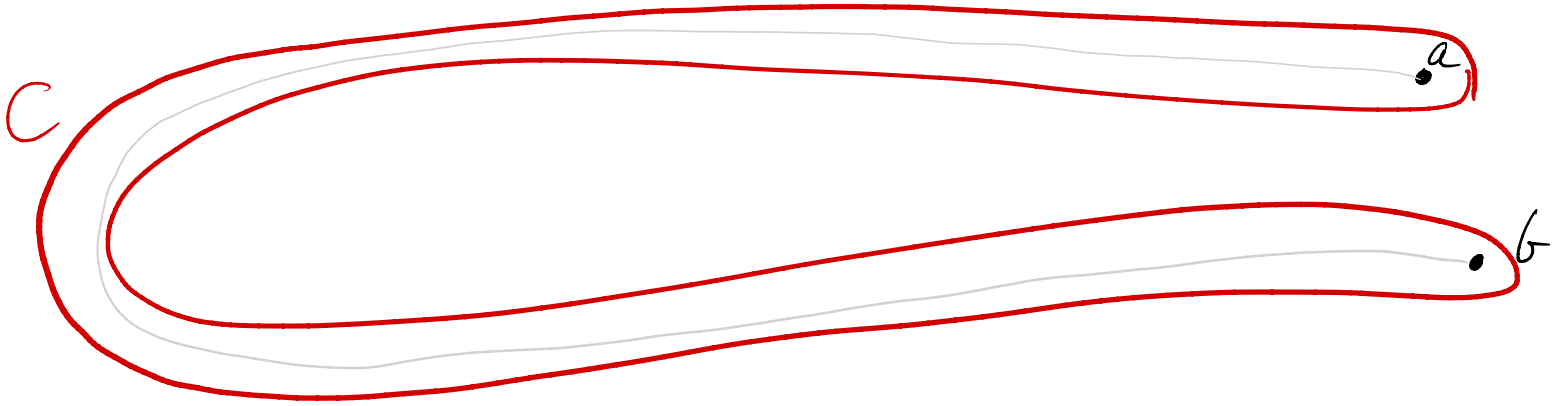
$$\|F(b) - F(a)\|_Y \leq k \|b - a\| \qquad \forall\, a, b \in C.$$

Proof: Let $G: [0,1] \to Y$, $G(t) = F(a + t(b-a))$. Then G is Fréchet differentiable with $G'(t) = F'(a+t(b-a))(b-a)$ (Exercise!).

G is continuous, and $\|G'(t)\|_Y \leq \|F'(a+t(b-a))\|_{\mathcal{L}} \|b-a\|_X \leq k\|b-a\|$.

Let $g(t) = tk\|b-a\|_X$ $(t \in [0,1])$ and apply the lemma. ▨

Note: The assumption of convexity is necessary:



$\|F'(x)\|_2$ could be small, but $\|F(b)-F(a)\|_y$ could be big, even if $\|b-a\|_x$ is small.

QUESTIONS?

COMMENTS?