PARTIAL DERIVATIVES

Definition

Let $X_1, \dots, X_n, Y$ be normed vector spaces, let $F: X_1 \times \dots \times X_n \longrightarrow Y$ be a function and fix $\bar{x} \in X_1 \times \dots \times X_n$.

A bounded linear operator $A \in \mathcal{L}(X_i, Y)$ is the i-th partial derivative of F at $\bar{x}$ if

$$\lim_{\substack{h \to 0 \\ h \in X_i}} \frac{\| F(\bar{x}_1, \dots, \bar{x}_i + h, \dots, \bar{x}_n) - F(\bar{x}) - Ah \|_Y}{\| h \|_{X_i}} = 0$$

Just as for Fréchet derivatives,

The i-th partial derivative of F at $\bar{x}$ is unique (if it exists).

We will call this bounded linear operator

$$\frac{\partial F}{\partial x_i}(\bar{x}) \in \mathcal{L}(X_i, Y).$$

Just as for functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$,

If $F : X_1 \times \cdots \times X_n \to Y$ is Fréchet differentiable at $\bar{x}$ then its partial derivatives exist, and

$$\frac{\partial F}{\partial x_i}(\bar{x})(h) = F'(\bar{x})(0, \ldots, h, \ldots 0) \qquad \forall\, h \in X_i$$

(h in the i-th position)

Conversely:

If all partial derivatives $\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}$ exist and are continuous at $\bar{x}$, then F is Fréchet differentiable at $\bar{x}$, and

$$F'(\bar{x})(r) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(\bar{x})(r_i)$$

Example:

Let $U = C([0,1])$ (equipped with the sup-norm $\|\cdot\|_\infty$) and define $F : U \times U \to U$ by

$$F(u,v)(t) = \int_0^t \left(u(s) + v(s)\right)^2 ds.$$

Compute $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$.

Example:

Let $U = C([0,1])$ (equipped with the sup-norm $\|\cdot\|_\infty$) and define $F : U \times U \to U$ by

$$F(u,v)(t) = \int_0^t (u(s)+v(s))^2 \, ds.$$

Compute $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$.

Initial guess: $\frac{\partial}{\partial u}(u+v)^2 = 2(u+v)$, so we might expect something like

$$\frac{\partial F}{\partial u}(u,v) = \int 2(u+v)\, ds$$

Compute first

$$F(u+h,v)(t) - F(u,v)(t) = \int_0^t \big(u(s)+h(s)+v(s)\big)^2 ds - \int_0^t \big(u(s)+v(s)\big)^2 ds$$

$$= \int_0^t 2\,h(s)\big(u(s)+v(s)\big) + h(s)^2\, ds.$$

We make a guess that $\frac{\partial F}{\partial u}(u,v) = A$, $\quad A(h)(t) = 2\int_0^t h(s)\big(u(s)+v(s)\big)\, ds.$

Let $\sigma(h) := F(u+h,v) - F(u,v) - A(h)$. Then

$$\frac{\|\sigma(h)\|_\infty}{\|h\|_\infty} = \frac{\sup_t \left| \int_0^t h(s)^2\, ds \right|}{\|h\|_\infty} \leq \frac{\sup_t \int_0^t \|h\|_\infty^2\, ds}{\|h\|_\infty}$$

$$= \|h\|_\infty \cdot \int_0^1 1\, ds = \|h\|_\infty \longrightarrow 0 \quad \text{as } h \to 0.$$

so if $A \in \mathcal{L}(U,U)$ then indeed $\frac{\partial F}{\partial u}(u,v) = A$.

Clearly, A is linear. Next,

$$\begin{aligned}
\|A(h)\|_\infty &= \sup_t \left| \int_0^t 2h(s)\,(u(s)+v(s))\,ds \right| \\
&\le 2 \sup_t \int_0^t |h(s)|\cdot|u(s)+v(s)|\,ds \\
&\le 2 \sup_t \int_0^t \|h\|_\infty \cdot \|u+v\|_\infty\,ds \\
&= 2\cdot\|h\|_\infty\cdot\|u+v\|_\infty
\end{aligned}$$

so $\|A(h)\|_\infty \le C\|h\|_\infty$, where $C = 2\|u+v\|_\infty \le 2\|u\|_\infty + 2\|v\|_\infty < \infty$.

We conclude that $A \in \mathcal{L}(U,U)$.

□