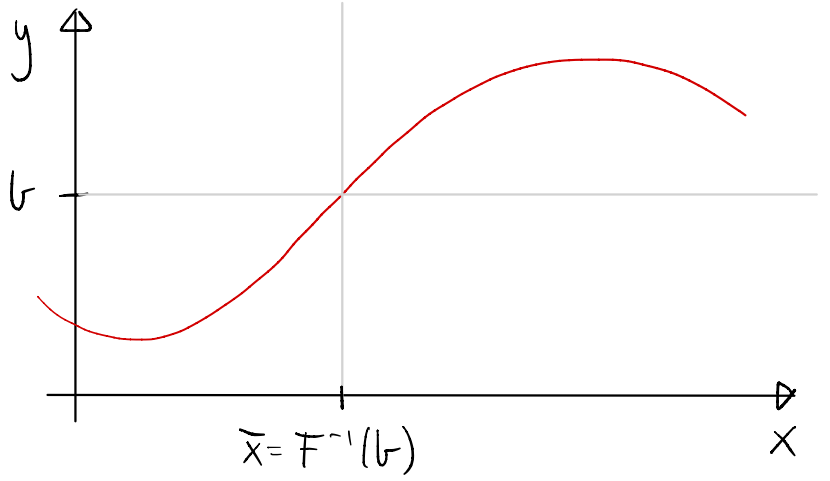
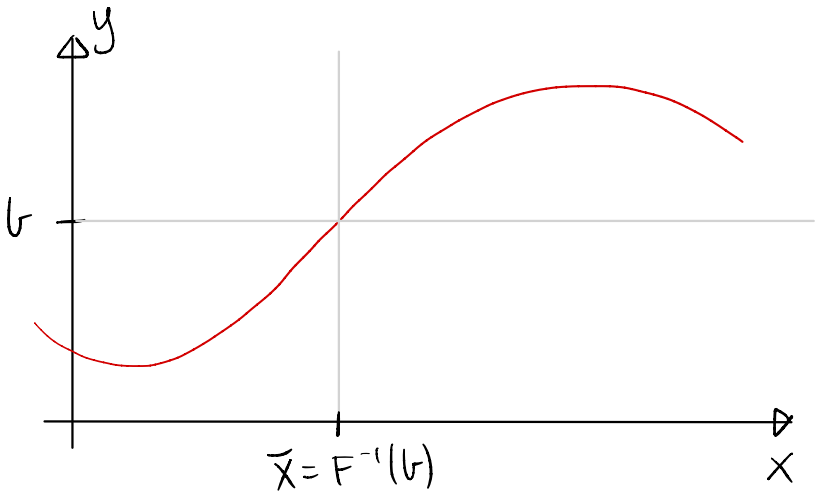
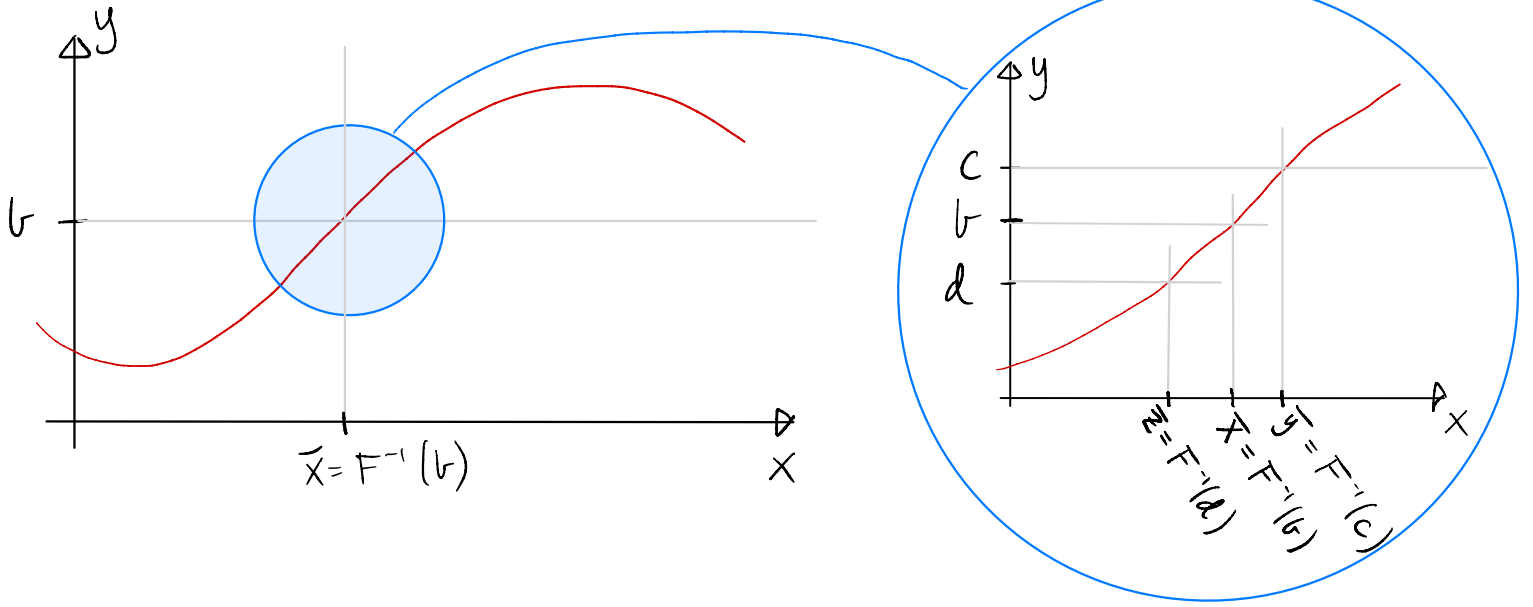
THE INVERSE FUNCTION THEOREM

We would like to solve the equation $F(\bar{x}) = b$ for some $F: X \to Y$ and $b \in Y$.

Then "$\bar{x} = F^{-1}(b)$". We will assume that the equation has a solution, and prove that the inverse F^{-1} also exists near b.

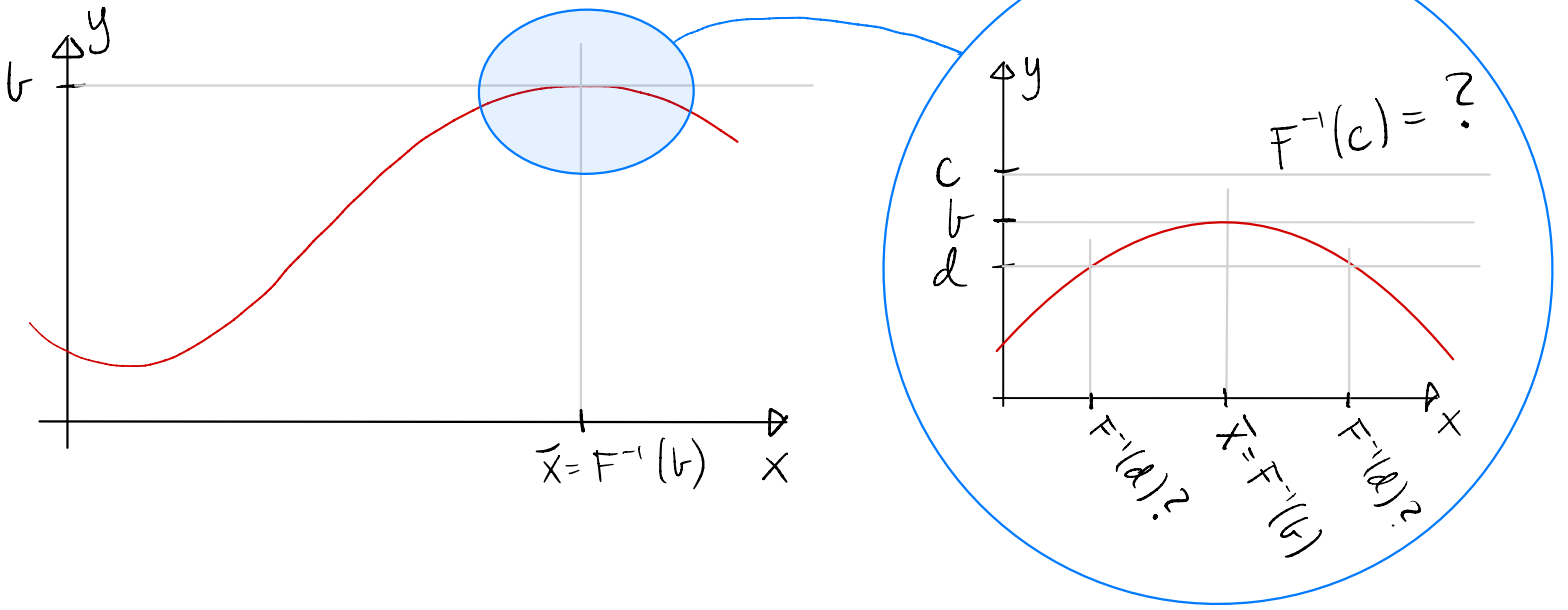



y
u
$\bar{x} = F^{-1}(u)$
x



Since $F'(\bar{x}) > 0$, there is a small neighborhood around $\bar{x}$ where $F'(x) > 0$, so F is <u>strictly increasing</u>, hence invertible there.

(The same would be true if $F'(\bar{x}) < 0$.)



If $F'(\bar{x}) = 0$ then F^{-1} is <u>not</u> well-defined near $\bar{x}$: it could attain zero, or many different values.

For $F: \mathbb{R} \to \mathbb{R}$, we need $F'(\bar{x}) \neq 0$ and continuity of F' for invertibility near $\bar{x}$.

Then $y = F(F^{-1}(y))$.

Differentiate on both sides:

$$1 = F'(F^{-1}(y))\,(F^{-1})'(y)$$

$$\Rightarrow \quad (F^{-1})'(y) = \frac{1}{F'(F^{-1}(y))}$$

so F' must be nonzero at $\bar{x}$ (and nearby points)!

If $F: X \to Y$ has an inverse $F^{-1}: Y \to X$ then:

$$y = F(F^{-1}(y)) \quad \forall y \iff I_y = F \circ F^{-1}$$

$$\Rightarrow y = I_y(y) = F'(F^{-1}(y))\left((F^{-1})'(y)\right) \quad \forall y$$

$$\Rightarrow \boxed{(F^{-1})'(y) = \left(F'(F^{-1}(y))\right)^{-1}}$$

Thus:

The Fréchet derivative F' must be continuous and $F'(\bar{x})$ must be invertible

Recall: $B \in \mathcal{L}(X, Y)$ is invertible if B is bijective and $B^{-1} \in \mathcal{L}(Y, X)$.

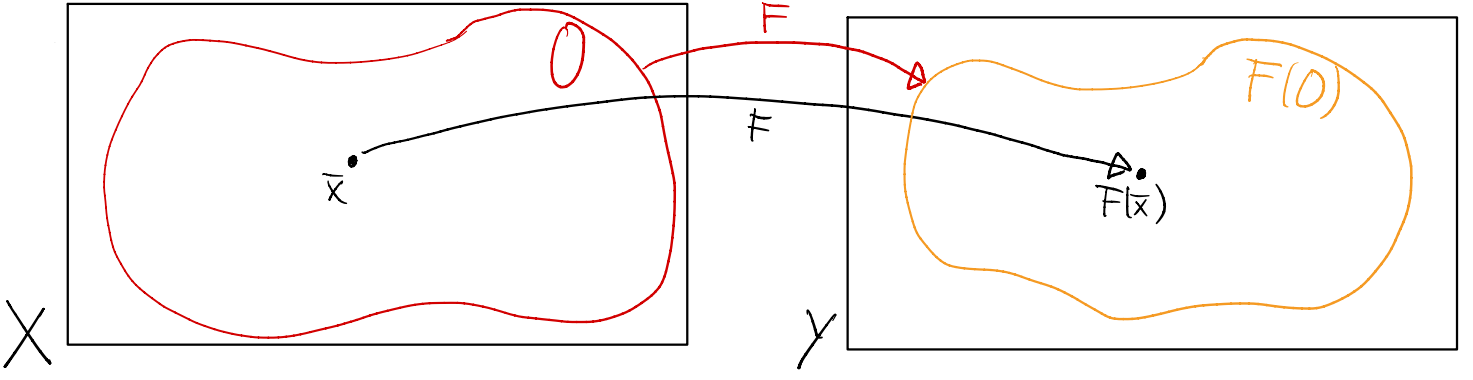
Note: F' is continuous at $\bar{x}$ if

$$\lim_{x \to \bar{x}} \|F'(x) - F'(\bar{x})\|_{\mathcal{L}} = 0.$$

The inverse function theorem

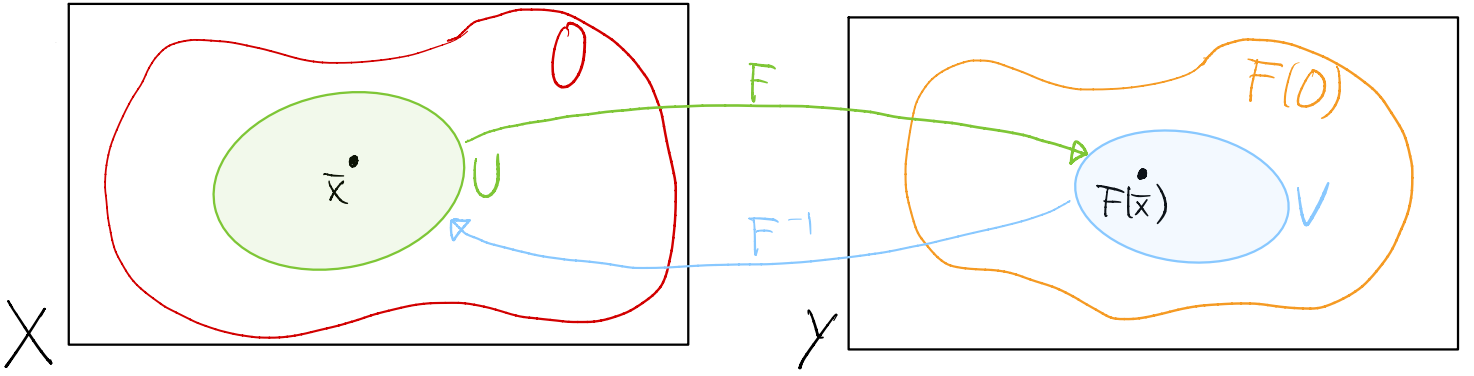
Let X, Y be normed vector spaces and X be complete. Let $O \subseteq X$ be open, $F: O \to Y$ be Fréchet differentiable and $\bar{x} \in O$ some point where F' is continuous and $F'(\bar{x})$ is invertible. Then there is a neighbourhood U of $\bar{x}$ and V of $F(\bar{x})$ such that $F: U \to V$ is bijective, F^{-1} is Fréchet differentiable at all $y \in V$, and

$$(F^{-1})'(y) = \left(F'(F^{-1}(y))\right)^{-1} \quad \forall\, y \in V.$$



The inverse function theorem

Let X, Y be normed vector spaces and X be complete. Let $O \subseteq X$ be open, $F: O \to Y$ be Fréchet differentiable and $\bar{x} \in O$ some point where F' is continuous and $F'(\bar{x})$ is invertible. Then there is a neighbourhood U of $\bar{x}$ and V of $F(\bar{x})$ such that $F: U \to V$ is bijective, F^{-1} is Fréchet differentiable at all $y \in V$, and $(F^{-1})'(y) = \left(F'(F^{-1}(y))\right)^{-1} \quad \forall\, y \in V.$



Main steps:

1) "Normalize" F so that $F'(\bar{x}) = I_X$. In particular, $\|\sigma(x)\|_Y / \|x-\bar{x}\| \xrightarrow[x\to\bar{x}]{} 0$, where $\sigma(x) = F(x) - F(\bar{x}) - x + \bar{x}$

2) Use Banach's lemma to prove that $F'(x)$ is invertible for all x close to $\bar{x}$

3) For any $y \in Y$ we want to find "$x = F^{-1}(y)$", i.e. some $x \in X$ so that $F(x) = y \iff x = \underbrace{y - F(\bar{x}) - \sigma(x) + \bar{x}}_{= H(x)}$

4) Prove that H is a contraction close to $\bar{x}$

5) Apply Banach's fixed point theorem to find x

6) Prove that F^{-1} is differentiable and compute its derivative

Proof of the inverse function theorem

Step 1: Define $G(x) = F'(\bar{x})^{-1}(F(x)) \iff F(x) = F'(\bar{x})(G(x))$

Then G is differentiable with $G'(x) = F'(\bar{x})^{-1} \circ F'(x)$.

Thus, G' is continuous at $\bar{x}$, since F' is continuous at $\bar{x}$ and $F'(\bar{x})$ is invertible. In particular, $G'(\bar{x}) = I_x$.

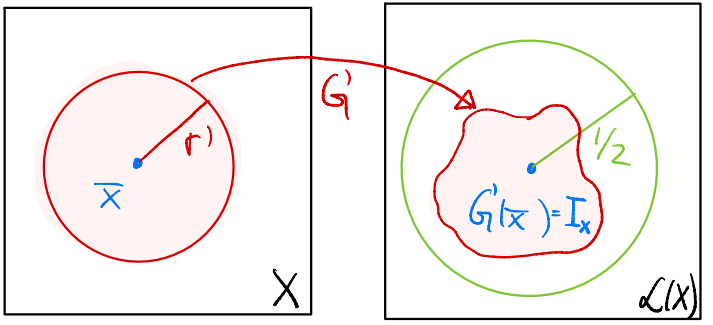
goal: Prove that G is bijective in some neighbourhood U of $\bar{x}$, and that $G^{-1}(F'(\bar{x})^{-1}(\bar{y})) = I_x$ (where $\bar{y} = F(\bar{x})$)

Then F is bijective ($F^{-1} = G^{-1} \circ (F'(\bar{x}))^{-1}$), and

$$(F^{-1})'(\bar{y}) = (G^{-1})'\left((F'(\bar{x}))^{-1}(\bar{y})\right)\left((F'(\bar{x}))^{-1}(\bar{y})\right) = (F'(\bar{x}))^{-1}(\bar{y}),$$

which is what we wanted.

Step 2:



G' is continuous at $\bar{x}$,

so there is some $r' > 0$ s.t.

$$\|\underbrace{G'(\bar{x})}_{= I_x} - G'(x)\|_{\mathcal{L}} < \frac{1}{2} \quad \forall\, x \in B(\bar{x}; r').$$

Banach's lemma then implies that

$G'(x)$ is invertible for all $x \in B(\bar{x}; r')$

and

$$\|G'(x)^{-1}\|_{\mathcal{L}} \leq \frac{1}{1 - \|I_x - G'(x)\|_{\mathcal{L}}} < 2$$

Step 3:

Fix some $y \in X$. We want to find "$x = G^{-1}(y)$" $\Leftrightarrow$ $G(x) = y$.

Let $\sigma(x) = G(x) - G(\bar{x}) - G'(\bar{x})(x - \bar{x}) = G(x) - G(\bar{x}) - x + \bar{x}$

and recall that $$\frac{\|\sigma(x)\|_X}{\|x - \bar{x}\|_X} \xrightarrow[x \to \bar{x}]{} 0.$$

If $G(x) = y$ then $\sigma(x) = y - G(\bar{x}) - x + \bar{x}$ $\Leftrightarrow$ $x = H(x)$,

where $H(x) = y - G(\bar{x}) + \bar{x} - \sigma(x)$. Thus,

we want to find a fixed point of H.

Step 4:

Since $\dfrac{\|\sigma(x)\|_X}{\|x-\bar{x}\|_X} \xrightarrow[x\to\bar{x}]{} 0$, there is some $r'' > 0$ such that $\|\sigma(x)\|_X \le \frac{1}{2}\|x-\bar{x}\|_X \quad \forall \; x \in \overline{B}(\bar{x}; r'')$.

Let $r = \min(r', r'')$ and $\tilde{U} = \overline{B}(\bar{x}; r)$.

Claim: $H(\tilde{U}) \subseteq \tilde{U}$, and H is a contraction on $\tilde{U}$.

- If $x \in \tilde{U}$ then $\|H(x)-\bar{x}\|_X = \|y - G(\bar{x}) - \sigma(x)\|_X \le \|y-G(\bar{x})\|_X + \|\sigma(x)\|_X$

  if $\|y-G(\bar{x})\|_X < \frac{1}{2}r$ (<) $\quad \frac{r}{2} + \frac{\|x-\bar{x}\|_X}{2} \le \frac{r}{2} + \frac{r}{2} = r$

  so $H(x) \in \tilde{U}$ if $y \in V = B(G(\bar{x}); r/2)$

- $\|H'(x)\|_{\mathcal{L}} = \|\sigma'(x)\|_{\mathcal{L}} = \|G'(x) - G'(\bar{x})\|_{\mathcal{L}} \le 1/2$

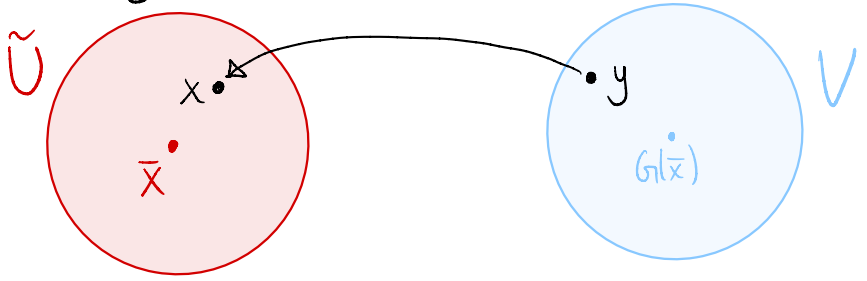
  so by the Mean Value Theorem, H is a contraction (with constant $1/2$).

Step 5: The set $\tilde{U} = \overline{B}(\bar{x}; r)$ is a closed subset of the complete space X, so it is a complete metric space.

Hence, we can apply Banach's fixed point theorem:

$$\exists!\ x \in \tilde{U} \text{ satisfying } x = H(x)$$
$$\Leftrightarrow \exists!\ x \in \tilde{U} \text{ satisfying } G(x) = y.$$

Thus: $\forall\ y \in V\ \exists!\ x \in \tilde{U}$ satisfying $G(x) = y$.

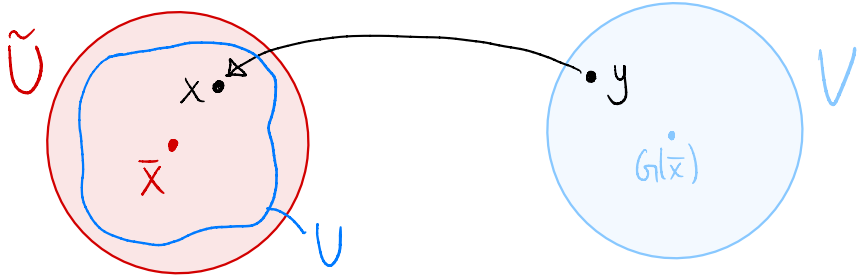


Denote this unique $x \in \tilde{U}$ by $G^{-1}(y)$. This defines an inverse function

$$G^{-1} : V \to \tilde{U}, \qquad G(G^{-1}(y)) = y \quad \forall\, y \in V.$$

Let $U = G^{-1}(V) \subseteq \tilde{U}$. Then:

- U is a neighborhood of $\bar{x}$ and V of $G(\bar{x})$
- $G : U \to V$ is bijective



Step 6: We claim that G^{-1} is differentiable, and

$$(G^{-1})'(y) = \left(G'(G^{-1}(y))\right)^{-1} \qquad \forall\ y \in V.$$

From Step 2 we know that

$G'(x)$ is invertible for all $x \in U$ and $\|G'(x)^{-1}\| \leq 2$

Moreover, G is differentiable at all $x \in U$, so if

$$\sigma_x(z) = G(z) - G(x) - G'(x)(z-x)$$

then $\|\sigma_x(z)\|_x \Big/ \|z-x\|_x \xrightarrow[z \to x]{} 0$ so there is some $\delta > 0$ such that $\|\sigma_x(z)\| \leq \frac{1}{4}\|z-x\|_x \quad \forall\ z \in B(x;\delta)$.

$\|\sigma_x(z)\| \leq \frac{1}{4}\|z-x\|_x \quad \forall\ z \in B(x;\delta)$, where $\sigma_x(z) = G(z) - G(x) - G'(x)(z-x)$

Hence, $x - z = G'(x)^{-1}\left(\sigma_x(z) + G(x) - G(z)\right)$

$$\Rightarrow \|x-z\|_x \leq \|G'(x)^{-1}\|_{\mathcal{L}} \|\sigma_x(z)\|_x + \|G'(x)^{-1}\|_{\mathcal{L}} \|G(x) - G(z)\|_x$$

$$\leq 2 \cdot \frac{1}{4} \cdot \|x-z\|_x + 2\|G(x) - G(z)\|_x$$

$$\Rightarrow \|x-z\|_x \leq 4\|G(x) - G(z)\|_x$$

For some $y \in V$, set $x = G^{-1}(y)$ and let $\tilde{y} = G(z)$:

$$\|x-z\|_x \leq 4\|\tilde{y} - y\|_x$$

and $G^{-1}(\tilde{y}) - G^{-1}(y) - G'(G^{-1}(y))^{-1}(\tilde{y}-y) = -G'(G^{-1}(y))^{-1}(\sigma_x(z))$

$$\Rightarrow \|G^{-1}(\tilde{y}) - G^{-1}(y) - G'(G^{-1}(y))^{-1}(\tilde{y}-y)\|_x \leq 2 \cdot \|\sigma_x(z)\|$$

- $x = G^{-1}(y), \quad \tilde{y} = G(z)$
- $\|x - z\|_X \leq 4 \|\tilde{y} - y\|_X$
- $\|G^{-1}(\tilde{y}) - G^{-1}(y) - G'(G^{-1}(y))^{-1}(\tilde{y} - y)\|_X \leq 2 \cdot \|\sigma_x(z)\|$

Finally,

$$\frac{\|G^{-1}(\tilde{y}) - G^{-1}(y) - G'(G^{-1}(y))^{-1}(\tilde{y} - y)\|_X}{\|y - \tilde{y}\|_X} \leq \frac{2\|\sigma_x(z)\|_X}{\frac{1}{4}\|x - z\|_X} = 8\frac{\|\sigma_x(z)\|_X}{\|x - z\|_X}$$

$$\longrightarrow 0$$

as $\tilde{y} \longrightarrow y \quad (\Longleftrightarrow z \longrightarrow x)$.

∎

QUESTIONS?

COMMENTS?