THE IMPLICIT FUNCTION THEOREM

I

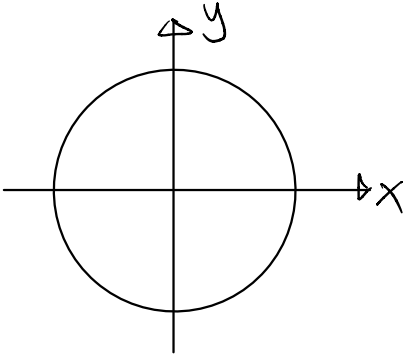
When solving an equation, there are often multiple variables, and we want to solve for one of them.

Example

Solve the equation $x^2+y^2=1$ for y as a function of x.

Rewrite the eq. as $F(x,y)=0$, where $F(x,y)=x^2+y^2-1$. We want to find some $G=G(x)$ such that

$$F(x, G(x)) = 0 \quad \forall\, x$$



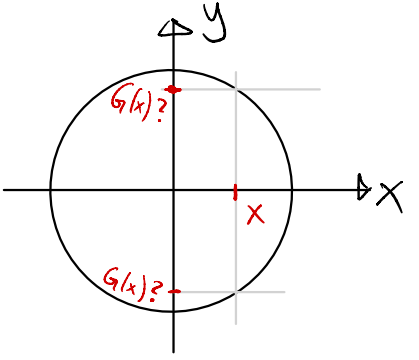
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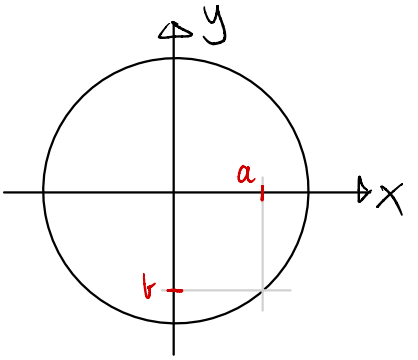
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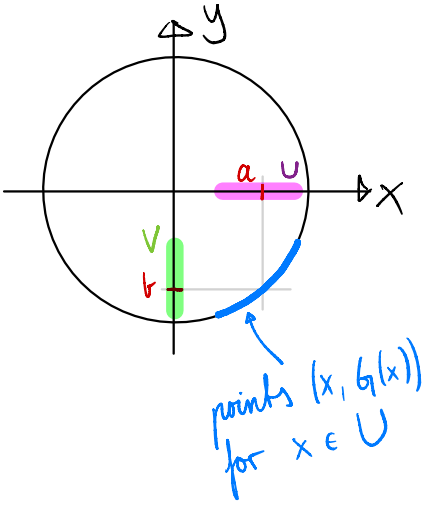
- It is clear that G cannot be unique!
- We need to specify <u>which part of the solution set</u> we want to solve for.

Let us fix a point (a, b) in the solution set: $F(a, b) = 0$.



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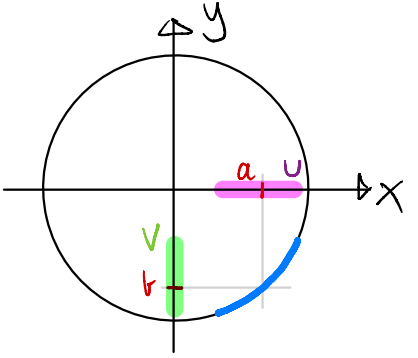
(*) $$F(x, G(x)) = 0 \qquad \forall\ x \in U$$

where U is a neighbourhood of a and V of b.

Differentiate (*) with respect to x:

$$\frac{\partial F}{\partial x}(x, G(x)) + \frac{\partial F}{\partial y}(x, G(x))\, G'(x) = 0$$

$$\Rightarrow \quad G'(x) = - \frac{\frac{\partial F}{\partial x}(x, G(x))}{\frac{\partial F}{\partial y}(x, G(x))}$$



$\frac{\partial F}{\partial y}$ must be nonzero at $(x, G(x))$ for all $x \in U$!

Generalize now to $F: X \times Y \to Z$: For some point $(a,b) \in X \times Y$ where $F(a,b) = 0$, we try to find nbd. U of a and V of b and some $G: U \to V$ s.t.

$$F(x, G(x)) = 0 \quad \forall\, x \in U.$$

Differentiate w.r.t. x:

$$0 = \frac{\partial F}{\partial x}(x, G(x)) + \frac{\partial F}{\partial y}(x, G(x)) \circ G'(x)$$

$$\Rightarrow \quad G'(x) = -\left(\frac{\partial F}{\partial y}(x, G(x))\right)^{-1} \circ \frac{\partial F}{\partial x}(x, G(x)).$$

$\frac{\partial F}{\partial y}(x, G(x))$ must be invertible for all $x \in U$!

QUESTIONS?
COMMENTS?

Next video: The implicit function theorem