THE IMPLICIT FUNCTION THEOREM

II

The implicit function theorem

Let X, Y, Z be Banach spaces, $O \subseteq X \times Y$ open, $F: O \to Z$ Fréchet differentiable and $(a,b) \in O$ a point where F' is continuous, $\frac{\partial F}{\partial y}$ is invertible, and where $F(a,b) = 0$. Then there exists open neighborhoods $U \subseteq X$ of a and $V \subseteq Y$ of b such that $U \times V \subseteq O$, and some Fréchet differentiable $G: U \to V$ such that

$$F(x, G(x)) = 0 \qquad \forall\ x \in U$$

and

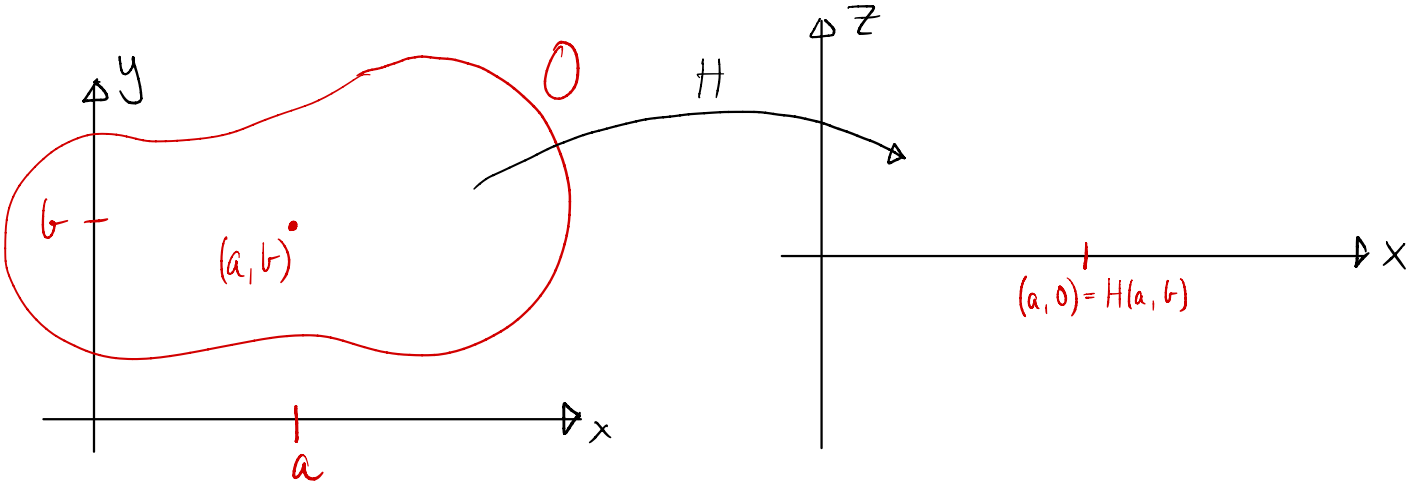
$$G'(x) = -\left(\frac{\partial F}{\partial y}(x, G(x))\right)^{-1} \circ \frac{\partial F}{\partial x}(x, G(x)) \qquad \forall\ x \in U.$$

Proof

Define $H: O \to X \times Z$ by $H(x,y) = (x, F(x,y))$. Then

- $H(a,b) = (a,0)$
- H is Fréchet differentiable
- $H'(x,y)(r,s) = \left(r, \frac{\partial F}{\partial x}(x,y)(r) + \frac{\partial F}{\partial y}(x,y)(s)\right) \quad \forall (r,s) \in X \times Y$

(exercise!)



- $H : O \longrightarrow X \times Z, \quad H(x,y) = (x, F(x,y))$
- $H(a,b) = (a,0)$
- $H'(x,y)(r,s) = \left(r, \frac{\partial F}{\partial x}(x,y)(r) + \frac{\partial F}{\partial y}(x,y)(s)\right) \quad \forall\, (r,s) \in X \times Y$

Claim: $H'(a,b)$ is invertible.

(i) $H'(a,b)$ is bijective: $H'(a,b)(r,s) = (u,v)$

$$\Leftrightarrow \quad \left(r, \frac{\partial F}{\partial x}(a,b)(r) + \frac{\partial F}{\partial y}(a,b)(s)\right) = (u,v)$$

$$\Leftrightarrow \quad (r,s) = \left(u, \frac{\partial F}{\partial y}(a,b)^{-1}\left(v - \frac{\partial F}{\partial x}(a,b)(u)\right)\right).$$

Thus: $H'(a,b)^{-1}(u,v) = \left(u, \frac{\partial F}{\partial y}(a,b)^{-1}\left(v - \frac{\partial F}{\partial x}(a,b)(u)\right)\right).$

(ii) $H'(a,b)^{-1}$ is bounded, since $\frac{\partial F}{\partial y}(a,b)^{-1}$ and $\frac{\partial F}{\partial x}(a,b)$ were assumed to be bounded.

- $H : O \longrightarrow X \times Z, \quad H(x,y) = (x, F(x,y))$
- $H(a,b) = (a,0)$
- $H'(x,y)(r,s) = \left(r, \frac{\partial F}{\partial x}(x,y)(r) + \frac{\partial F}{\partial y}(x,y)(s)\right) \quad \forall (r,s) \in X \times Y$
- $H'(a,b)$ is invertible.

Finally, H' is continuous at (a,b), since $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are.

By the inverse function theorem, there is some open set $W \subseteq O$ containing (a,b) such that H is bijective on W, H^{-1} is Frechet differentiable, and

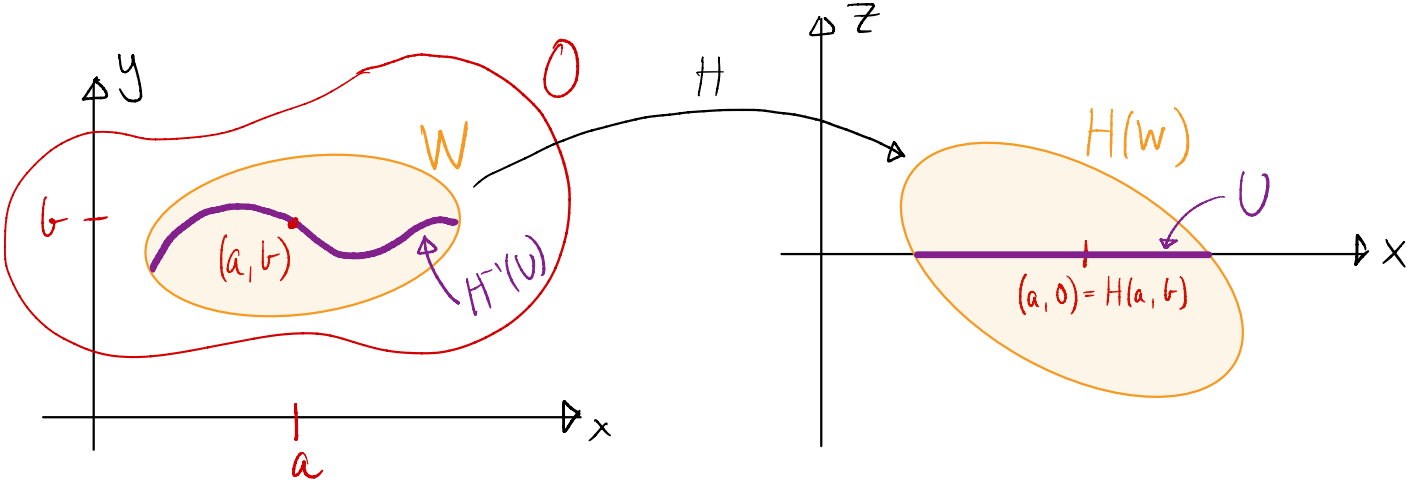
$$(H^{-1})'(H(x,y)) = (H'(x,y))^{-1} \qquad \forall \ (x,y) \in W.$$

Define the open set $U = \{x \in X : (x, 0) \in H(W)\}$.

Since $(a, b) \in W$ and $H(a, b) = (a, 0)$, we have $a \in U$.

For each $x \in U$, let $G(x)$ be defined by $(x, G(x)) = H^{-1}(x, 0)$

$\iff H(x, G(x)) = (x, 0) \iff F(x, G(x)) = 0$.



Then G is Fréchet differentiable:

$(x, G(x)) = H^{-1}(x, 0)$ and $H'(a,b)^{-1}(u,v) = \left(u, \frac{\partial F}{\partial y}(a,b)^{-1}\left(v - \frac{\partial F}{\partial x}(a,b)(u)\right)\right)$

$$\Rightarrow \left(u, G'(a)(u)\right) = \left(u, \frac{\partial F}{\partial y}(a,b)^{-1}\left(0 - \frac{\partial F}{\partial x}(a,b)(u)\right)\right)$$

$$= \left(u, -\frac{\partial F}{\partial y}(a,b)^{-1} \circ \frac{\partial F}{\partial x}(a,b)\,(u)\right)$$

$$\Rightarrow \quad G'(a) = -\frac{\partial F}{\partial y}(a,b)^{-1} \circ \frac{\partial F}{\partial x}(a,b).$$



Corollary:

Let $O \subseteq \mathbb{R}^{m \times n}$ be open, $F : O \to \mathbb{R}^n$ Fréchet differentiable, and $(a, b) \in O$ (where $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$) a point where $\frac{\partial F}{\partial y}$ is invertible, F' is continuous and $F(a, b) = 0$.

Then there is an open set $U \subseteq \mathbb{R}^m$ and a differentiable $G : U \to \mathbb{R}^n$ so that $F(x, G(x)) = 0 \quad \forall x \in U$ and

$$G'(a)(u) = -\left(J_y F(a, b)\right)^{-1} \cdot J_x F(a, b) \cdot u,$$

where $$J_y F(a, b) = \begin{pmatrix} \frac{\partial F_1}{\partial y_1}(a, b) & \cdots & \frac{\partial F_1}{\partial y_n}(a, b) \\ \vdots & \ddots & \\ \frac{\partial F_n}{\partial y_1}(a, b) & \cdots & \frac{\partial F_n}{\partial y_n}(a, b) \end{pmatrix}$$

QUESTIONS?
COMMENTS?