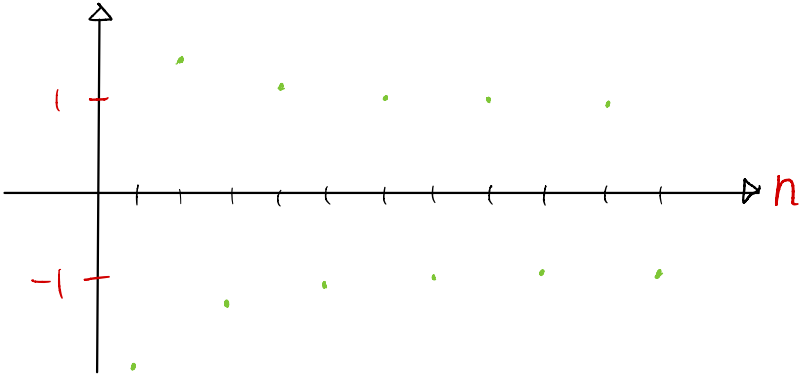
LIM INF AND LIM SUP

Let $x_n = (-1)^n \left(1 + \frac{1}{n}\right)$ for $n \in \mathbb{N}$.



The sequence does not converge.
But can we still extract information about the "upper/lower limits" of $\{x_n\}_n$, as $n \to \infty$?

Intuitively, $\limsup_{n\to\infty} x_n$ = (the largest value far into the sequence)

$\liminf_{n\to\infty} x_n$ = (the smallest value far into the sequence)

A sequence $\{x_n\}_n$ in $\mathbb{R}$ is

- <u>increasing</u> if $x_n \leq x_{n+1} \quad \forall n \in \mathbb{N}$
- <u>decreasing</u> if $x_n \geq x_{n+1} \quad \forall n \in \mathbb{N}$
- <u>monotone</u> if it is either increasing or decreasing.

A sequence $\{x_n\}_n$ in $\mathbb{R}$ is <u>bounded</u> if there is some $M > 0$ such that $|x_n| \leq M \quad \forall n \in \mathbb{N}$.

Example:

Let $\{x_n\}_n$ be any bounded sequence in $\mathbb{R}$. Let

$$y_n = \sup\{x_n, x_{n+1}, \dots\}$$

$$z_n = \inf\{x_n, x_{n+1}, \dots\}.$$

Then $\{y_n\}_n$ and $\{z_n\}_n$ are bounded, monotone sequences.

Every monotone, bounded sequence in $\mathbb{R}$ converges.

Proof: Let $\{x_n\}_n$ be an increasing, bounded sequence. Let

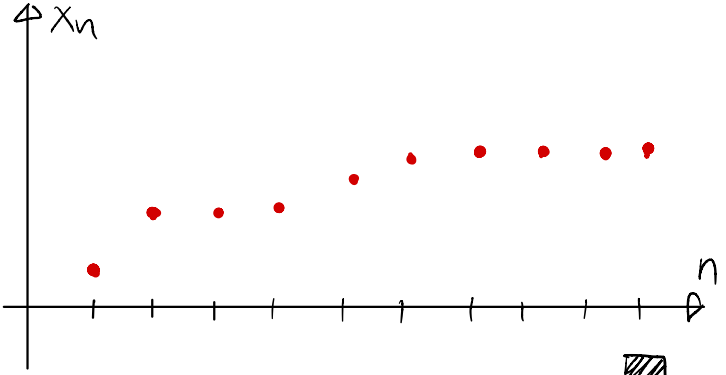
$$x = \sup A, \quad \text{where } A = \{x_1, x_2, \dots\}$$

Then $x < \infty$.

Let $\varepsilon > 0$. Then there is some $N \in \mathbb{N}$ such that $|x - x_N| < \varepsilon$, otherwise $x - \varepsilon$ would be a smaller upper bound for A.

If $n \geq N$ then also

$$|x - x_n| = x - x_n \leq x - x_N < \varepsilon.$$



∎

Let $\{x_n\}_n$ be a sequence in $\mathbb{R}$. Define

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup \{x_n, x_{n+1}, \dots\} \right)$$

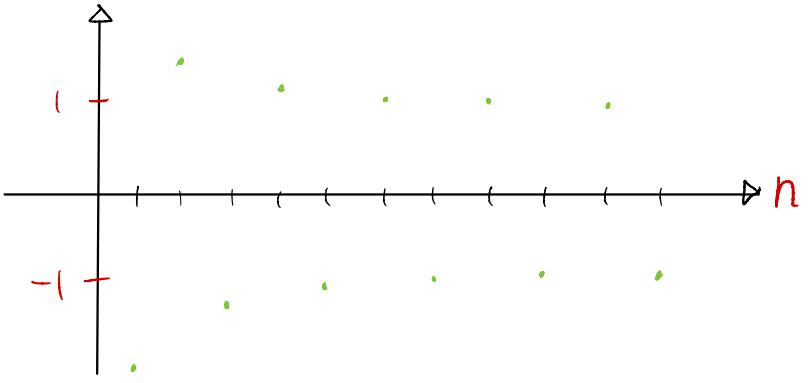
$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left(\inf \{x_n, x_{n+1}, \dots\} \right)$$

- If $\{x_n\}_n$ is not upper bounded then $\limsup_{n\to\infty} x_n = \infty$
- If $\{x_n\}_n$ is upper bounded then the sequence $\{y_n\}_n$ $(y_n = \sup \{x_n, x_{n+1}, \dots\})$ is a bounded, decreasing sequence, hence it has a limit.

Example

Let $x_n = (-1)^n \left(1 + 1/n\right)$ for $n \in \mathbb{N}$.

Compute $\limsup_n x_n$ and $\liminf_n x_n$.



$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup \{x_n, x_{n+1}, \dots\} \right)$$

$$= \lim_{n\to\infty} \left(\max \left((-1)^n \left(1 + 1/n\right), (-1)^{n+1} \left(1 + 1/(n+1)\right) \right) \right) = 1,$$

and in the same way we get $\liminf_{n\to\infty} = -1$.

Let $\{x_n\}_n$ and $\{y_n\}_n$ be sequences in $\mathbb{R}$ and assume $y_n \xrightarrow[n\to+\infty]{} \alpha$. Then:

(i) $$\limsup_{n\to+\infty} (x_n+y_n) = \left(\limsup_{n\to+\infty} x_n\right) + \alpha$$

$$\liminf_{n\to+\infty} (x_n+y_n) = \left(\liminf_{n\to+\infty} x_n\right) + \alpha$$

(ii) if also $\alpha > 0$ then

$$\limsup_{n\to+\infty} (x_n y_n) = \left(\limsup_{n\to+\infty} x_n\right)\alpha$$

$$\liminf_{n\to+\infty} (x y_n) = \left(\liminf_{n\to+\infty} x_n\right)\alpha$$

QUESTIONS?

COMMENTS?