MULTIINDICES

If $f: \mathbb{R} \to \mathbb{R}$ is "smooth enough" (differentiable as many times as we need) then we can approximate it by its Taylor expansion:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k + R_n(x)$$

What about $f: \mathbb{R}^d \to \mathbb{R}$, $d > 1$?

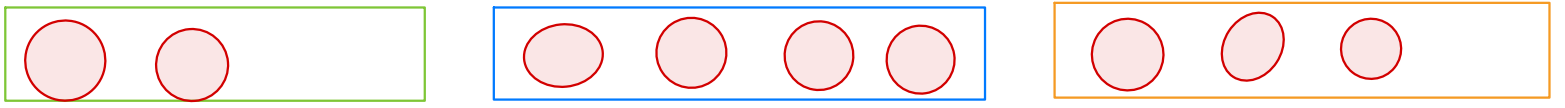
Fix $d \in \mathbb{N}$. Recall: $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

Def: A multiindex is a vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}_0^d$.

- The length of α is $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$
- The factorial of α is $\alpha! = \alpha_1! \, \alpha_2! \cdots \alpha_d!$
- The multinomial coefficient $\binom{k}{\alpha}$ (where $k = |\alpha|$) is

$$\binom{k}{\alpha} = \frac{k!}{\alpha!}$$

Consider d containers, with container j containing α_j objects.



$\alpha_1 = 2$ $\alpha_2 = 4$ $\alpha_3 = 3$

- $|\alpha|$ is the total number of objects.
- $\alpha!$ is the number of ways in which these $|\alpha|$ objects can be ordered without removing them from their container.
- $\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!}$ is the number of ways $|\alpha|$ objects can be placed in d containers, putting α_j objects in container j, not caring about their order.

If $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}_0^d$ then we write

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$$

(with the convention that $z^0 = 1 \quad \forall \ z \in \mathbb{R}$).

A <u>multivariate polynomial</u> is a function of the form

$$p(x) = \sum_{|\alpha| \leq k} c_\alpha x^\alpha \qquad \text{for } x \in \mathbb{R}^d$$

where $c_\alpha \in \mathbb{R}$ and the sum goes over all multindices of length $\leq k$ (the <u>order</u> of the polynomial).

Examples:

- $p(x) = 1 + 2x + 3x^2$ for $x \in \mathbb{R}$
- $p(x_1, x_2) = a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1 x_2$
- $p(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$

The multinomial theorem: If $x \in \mathbb{R}^d$ and $k \in \mathbb{N}$ then

$$(x_1 + x_2 + \cdots + x_d)^k = \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| = k}} \binom{k}{\alpha} x^\alpha$$

Proof:

$d=1$: $$\sum_{|\alpha|=k} \binom{k}{\alpha} x^\alpha = \sum_{\alpha_1 = k} \frac{k!}{\alpha_1!} x^{\alpha_1} = x_1^k$$ ✓

$d=2$: $$\sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha = \sum_{\substack{\alpha_1, \alpha_2 \in \mathbb{N}_0 \\ \alpha_1 + \alpha_2 = k}} \frac{k!}{\alpha_1!\,\alpha_2!} x_1^{\alpha_1} x_2^{\alpha_2} = \sum_{\ell=0}^{k} \frac{k!}{\ell!(k-\ell)!} x_1^\ell x_2^{k-\ell}$$

$$= (x_1 + x_2)^k \quad \text{(by the binomial theorem)}$$ ✓

$d \rightsquigarrow d+1$: Assume $(x_1+\cdots+x_d)^k = \sum_{\alpha\in\mathbb{N}_0^d : |\alpha|=k} \binom{k}{\alpha} x^\alpha$ for all $x\in\mathbb{R}^d$.

Then

$$\left(x_1+\cdots+x_d+x_{d+1}\right)^k = \left(\left(x_1+\cdots+x_d\right)+x_{d+1}\right)^k = \sum_{\ell=0}^{k} \frac{k!}{\ell!(k-\ell)!} \left(x_1+\cdots+x_d\right)^\ell x_{d+1}^{k-\ell}$$

$$= \sum_{\ell=0}^{k} \frac{k!}{\ell!(k-\ell)!} \sum_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|=\ell}} \frac{\ell!}{\alpha_1!\cdots\alpha_d!} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d} x_{d+1}^{k-\ell}$$

$$= \sum_{\ell=0}^{k} \sum_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|=\ell}} \frac{k!}{\alpha_1!\cdots\alpha_d!(k-\ell)!} x_1^{\alpha_1}\cdots x_d^{\alpha_d} x_{d+1}^{k-\ell}$$

$\beta = (\alpha_1,\ldots,\alpha_d,k-\ell) \in \mathbb{N}_0^{d+1}$

$$= \sum_{\substack{\beta\in\mathbb{N}_0^{d+1} \\ |\beta|=k}} \binom{k}{\beta} x^\beta$$

✓

■

Solve exercises 1, 2, 5 in the note on multiindices!

QUESTIONS?

COMMENTS?