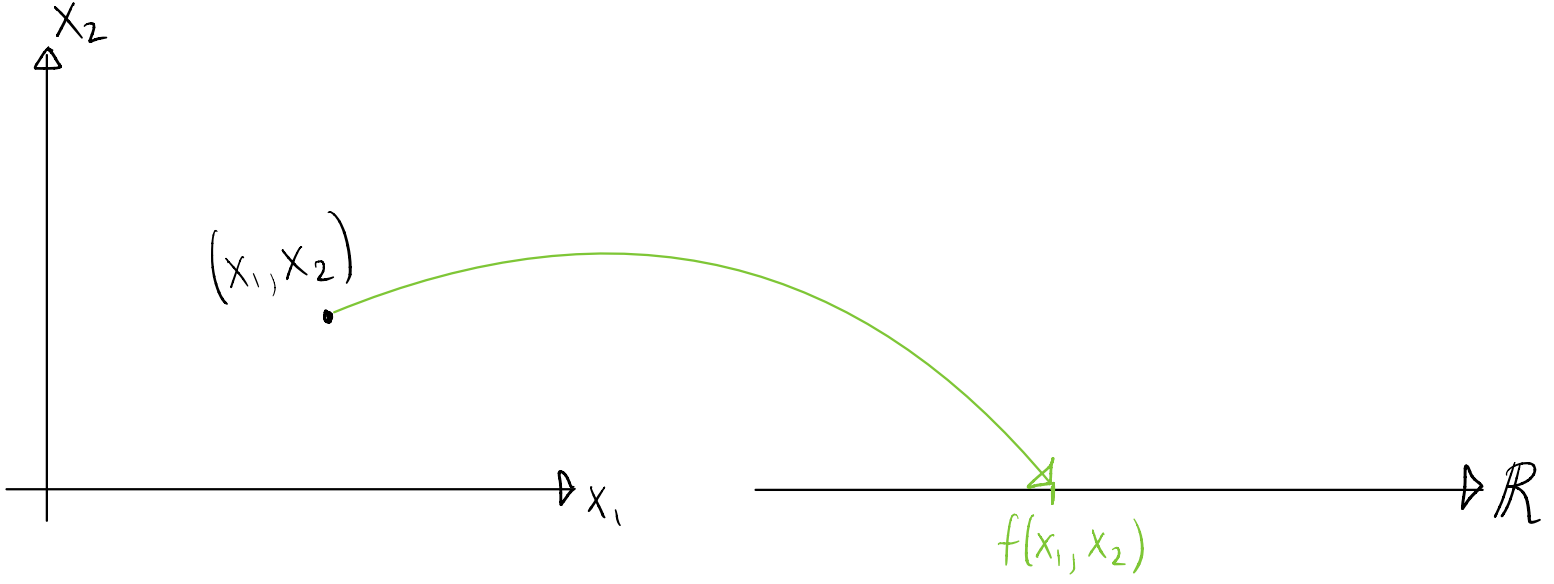
MULTIINDICES, PARTIAL DIFFERENTIATION AND TAYLOR'S THEOREM

Goal: Taylor's theorem for $f : \mathbb{R}^d \longrightarrow \mathbb{R}$



A (mixed) partial derivative is a function such as

$$\frac{\partial^3 f}{\partial x_1^2 \partial x_3}(x_1, x_2, x_3).$$

The α-th partial derivative of $f : \mathbb{R}^d \to \mathbb{R}$ is

$$f^{(\alpha)}(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}}(x)$$

Example: $\frac{\partial^3 f}{\partial x_1^2 \partial x_3} = f^{(\alpha)}$ for $\alpha = (2, 0, 1)$.

Leibniz' rule If f, g are functions and α a multiindex then

$$(fg)^{(\alpha)} = \sum_{\substack{\beta \in \mathbb{N}_0^d \\ \beta \leq \alpha}} \binom{\alpha}{\beta} f^{(\beta)} g^{(\alpha-\beta)}$$

where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$

Here, $\beta \leq \alpha$ means $\beta_1 \leq \alpha_1, \dots, \beta_d \leq \alpha_d$.

Idea of proof: Induction on d. When $d=1$ we have

$$(fg)^{(\alpha_1)} = \sum_{\beta_1=0}^{\alpha_1} \binom{\alpha_1}{\beta_1} f^{(\beta_1)} g^{(\alpha_1-\beta_1)}$$

(Leibniz' rule in one dimension). ∎

Taylor's theorem

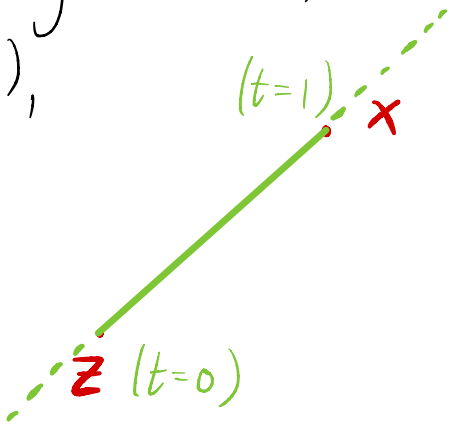
Let $f: \mathbb{R}^d \to \mathbb{R}$ be smooth and fix $z \in \mathbb{R}^d$, $k \in \mathbb{N}$. Then

$$f(x) = \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \le k}} \frac{1}{\alpha!} f^{(\alpha)}(z)(x-z)^\alpha + R(x)$$

where $|R(x)| \le C\|x-z\|^{k+1}$.

Proof: Assume $z = 0$ for simplicity. Define $g(t) = f(tx)$ for $t \in [0,1]$. Then $g(0) = f(0)$, $g(1) = f(x)$, and

$$f(x) = g(1) = \sum_{\ell=0}^{k} \frac{1}{\ell!} g^{(\ell)}(0) + R(1)$$



We compute

$$g'(t) = \sum_{n=1}^{d} \frac{\partial f}{\partial x_n}(tx)\, x_n$$

$$g''(t) = \sum_{m=1}^{d} \sum_{n=1}^{d} \frac{\partial^2 f}{\partial x_m \partial x_n}(tx)\, x_m x_n$$

$$\vdots$$

$$g^{(\ell)}(t) = \sum_{n_1=1}^{d} \cdots \sum_{n_\ell=1}^{d} \frac{\partial^\ell f}{\partial x_{n_1} \cdots \partial x_{n_\ell}}(tx)\, x_{n_1} \cdots x_{n_\ell}.$$

Each term $\frac{\partial^\ell f}{\partial x_{n_1} \cdots \partial x_{n_\ell}}$ can be written as $f^{(\alpha)}$ for some multiindex α with $|\alpha| = \ell$. The number of ways in which $f^{(\alpha)}$ can appear in this way is $\binom{\ell}{\alpha}$.

Thus: $$g^{(\ell)}(t) = \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| = \ell}} \binom{\ell}{\alpha} f^{(\alpha)}(t)\, x^{\alpha}$$

We get $$g(1) = \sum_{\ell=0}^{k} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| = \ell}} \frac{1}{\ell!} \binom{\ell}{\alpha} f^{(\alpha)}(0)\, x^{\alpha} + R(1)$$

$$= \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \le k}} \frac{1}{\ell!} \binom{|\alpha|}{\alpha} f^{(\alpha)}(0)\, x^{\alpha} + R(1)$$

where R is Taylor's remainder term.

We have $R(t) = \frac{t^{k+1}}{(k+1)!} g^{(k+1)}(\xi)$ for some ξ between 0 and t

$$\Rightarrow |R(1)| = \left| \frac{1}{(k+1)!} \sum_{|\alpha|=k+1} \binom{k+1}{\alpha} f^{(\alpha)}(\xi x)(\xi x)^{\alpha} \right|$$

$$\leq \frac{C}{(k+1)!} \cdot (k+1)! \sum_{|\alpha|=k+1} \frac{1}{\alpha!} |x_1|^{\alpha_1} \cdots |x_d|^{\alpha_d}$$

where $C \geq |f^{(\alpha)}(\xi x)|$ for all ξ, x, α

$$\leq C \|x\|^{k+1} \sum_{|\alpha|=k+1} \frac{1}{\alpha!}$$

$$= \frac{C d^k}{(k+1)!} \|x - z\|^{k+1}.$$

∎

Exercise: Compute the second-order Taylor expansion of

$$f(x,y) = e^{x\cos y - 1}$$

QUESTIONS?

COMMENTS?