UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT2400 — Real Analysis

Day of examination: 9 June 2021

Examination hours: 15:00 – 19:00

This problem set consists of 2 pages.

Appendices: None

Permitted aids: Any

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: There are in total 10 sub-problems, and you can get up to 10 points for each sub-problem, for a total of 100 points.

Problem 1. (10 points)
Let (X, d) be the metric space $X = (0, 1]$, $d(x, y) = |x - y|$, and let $T\colon X \to X$ be given by $T(x) = x/2$.

Show that T is a contraction. Does T have a fixed point? Justify your answer.

Problem 2. (10 points)
Let (X, d) be a metric space and let $A \subseteq \mathbb{R}$ be closed. We define the metric space $C_b(X, A) = \{\text{all continuous, bounded } f\colon X \to A\}$, equipped with the supremum metric

$$\rho(f, g) = \sup_{x \in X} |f(x) - g(x)|. \tag{1}$$

(You do not need to show that this is a metric space.) Show that $C_b(X, A)$ is a closed subset of $C_b(X, \mathbb{R})$.

Problem 3. (20 points)
Let (X, d) be a metric space, and for every nonempty $E \subseteq X$ and $x \in X$, define

$$\operatorname{dist}(x, E) = \inf\{d(x, y) : y \in E\}. \tag{2}$$

(a) Show that if E is compact and nonempty, then there is some $z \in E$ such that $\operatorname{dist}(x, E) = d(x, z)$.

(b) Give an example of a metric space (X, d), a point $x \in X$ and a nonempty subset $E \subseteq X$ for which there is no such point $z \in E$.

Problem 4. (20 points)
For this problem, recall that a bounded linear operator A is *invertible* if it is bijective and its inverse A^{-1} is bounded.

Let $(X, \|\cdot\|)$ be a normed vector space and let $A\colon X \to X$ be an invertible bounded linear operator. Define $\|x\|_A = \|Ax\|$ for every $x \in X$.

(Continued on page 2.)

(a) Show that $\| \cdot \|_A$ is a norm on X.

(b) Show that a sequence $\{x_n\}_n$ in X converges in the norm $\| \cdot \|$ if and only if it converges in the norm $\| \cdot \|_A$.

Problem 5. (10 points)
Let f be given by the series

$$f(x) = \sum_{n=1}^{\infty} n(x-1)^n, \qquad x \in \mathbb{R}. \tag{3}$$

Determine the set $D = \{x \in \mathbb{R} : f(x) \text{ converges}\}$. Compute the derivative f', and determine the corresponding set D' of points where the series for f' converges.

Problem 6. (10 points)
Let $f, g \colon [-\pi, \pi] \to \mathbb{C}$ be continuous functions satisfying

$$\int_{-\pi}^{\pi} f(x)e^{inx}\, dx = \int_{-\pi}^{\pi} g(x)e^{inx}\, dx \qquad \forall\, n \in \mathbb{Z}. \tag{4}$$

Show that $f = g$.

Problem 7. (20 points)
Let $X = C_b(\mathbb{R}, \mathbb{R})$, equipped with the supremum norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} |f(t)|$. Define

$$F \colon X \to X, \qquad F(f)(t) = 2f(t)^2 - e^{f(t)-t^2} \qquad \forall\, t \in \mathbb{R}. \tag{5}$$

(a) Prove that F is Fréchet differentiable and show that $F'(f) = A$ for $f \in X$, where $A \colon X \to X$ is given by

$$A(r)(t) = 4r(t)f(t) - r(t)e^{f(t)-t^2} \qquad \forall\, t \in \mathbb{R},\ r \in X. \tag{6}$$

Hint: You might need the fact that $|e^s - 1 - s| \leqslant \frac{e}{2}s^2$ *for every number* $|s| \leqslant 1$*. This follows from Taylor expansion of the exponential function.*

(b) Let $\mathbb{1} \colon \mathbb{R} \to \mathbb{R}$ be the constant function $\mathbb{1}(t) = 1$ for all $t \in \mathbb{R}$. Prove that F is bijective in a neighbourhood of $\mathbb{1}$. Compute $\big(F^{-1}\big)'(F(\mathbb{1}))$.