UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT2400 — Real Analysis
Day of examination:	18 August 2021
Examination hours:	09:00 – 13:00

This problem set consists of 2 pages.

Appendices:	None
Permitted aids:	Any

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: There are in total 10 sub-problems, and you can get up to 10 points for each sub-problem, for a total of 100 points.

Problem 1. (10 points)
Consider $\mathbb{R}^2$ equipped with the Euclidean norm $\|x\| = \sqrt{x_1^2 + x_2^2}$ and the corresponding metric $d(x, y) = \|x - y\|$. Define the unit circle $S = \{x \in \mathbb{R}^2 : \|x\| = 1\}$. Explain why

- (S, d) is a metric space
- $(S, \|\cdot\|)$ is *not* a normed vector space.

Note: You may use the fact that $(\mathbb{R}^2, \|\cdot\|)$ is a normed vector space.

Problem 2. (10 points)
Prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{-x^2}$ is uniformly continuous.

Problem 3. (20 points)
Fix some $p \in [1, \infty]$ and let $x \in \ell^p(\mathbb{R})$ with components $x = (x(1), x(2), \dots)$. Define $y_n = \big(x(1), x(2), \dots, x(n), 0, 0, 0, \dots\big)$ for every $n \in \mathbb{N}$, so that

$$y_1 = \big(x(1), 0, 0, 0, \dots\big), \quad y_2 = \big(x(1), x(2), 0, 0, 0, \dots\big),$$

and so on.

(a) Explain why $y_n \in \ell^p(\mathbb{R})$ for every $n \in \mathbb{N}$. Show that if $p < \infty$ then $\{y_n\}_{n\in\mathbb{N}}$ is Cauchy.

(b) Is the same true for $p = \infty$? If yes, prove it. If no, provide a counterexample.

Problem 4. (10 points)
Let $X = C([0, 1], \mathbb{R})$ be equipped with the supremum metric $d_\infty(f, g) = \sup_{t\in[0,1]} |f(t) - g(t)|$. Let $L\colon X \to \mathbb{R}$ be a continuous function satisfying

$$L(p) = 0 \qquad \text{for all polynomials } p.$$

(Continued on page 2.)

Show that $L(f) = 0$ for all $f \in X$.

Problem 5. (20 points)
Let $X = C([0,2],\mathbb{R})$, equipped with the supremum norm $\|f\|_\infty = \sup_{t\in[0,2]} |f(t)|$, and let $Y = C([0,1],\mathbb{R})$, equipped with the L^1 norm $\|g\|_1 = \int_0^1 |g(t)|\,dt$.

(a) Show that $L\colon X \to Y$, defined by

$$L(f) = g, \qquad \text{where } g(t) = f(2t)\ \forall t \in [0,1],$$

is a bounded linear functional.

(b) Show that L is bijective, but that the inverse $L^{-1}\colon Y \to X$ is unbounded.

Problem 6. (20 points)
Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$F(x) = \begin{pmatrix} x_1^3(x_2+1) - x_2^2 \\ x_1 - x_2^5 \end{pmatrix} \qquad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2.$$

(a) Show that F is everywhere Fréchet differentiable, and that

$$F'(x)(h) = \begin{pmatrix} 3x_1^2(x_2+1) & x_1^3 - 2x_2 \\ 1 & -5x_2^4 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \qquad \forall x, h \in \mathbb{R}^2.$$

(b)

(i) It's difficult, or impossible, to find a formula for F^{-1} (if it exists at all). Why?

(ii) Let

$$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and note that $F(b) = c$. Show that there is a neighbourhood U of b and a neighbourhood V of c such that $F\colon U \to V$ is bijective. Compute $(F^{-1})'(c)$.

Problem 7. (10 points)
A bounded linear operator $L\colon X \to Y$ is *compact* if $\{L(u_n)\}_{n\in\mathbb{N}}$ has a convergent subsequence whenever $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in X.

Let $X = Y = C([-\pi,\pi],\mathbb{R})$ equipped with the supremum metric, and for some fixed $N \in \mathbb{N}$, let L be the Fourier projection

$$L(u)(t) = \sum_{n=-N}^{N} \alpha_n e^{int} \qquad \forall t \in [-\pi,\pi],\ u \in X$$

where $\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(s)e^{-ins}\,ds$ is the nth Fourier coefficient of u. Show that L is a bounded, compact operator. (You do not need to show that L is linear.)

Hint: Apply the Arzela–Ascoli theorem.