UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT2400 — Real analysis
Day of examination:	Wednesday, June 1st, 2022
Examination hours:	15.00 – 19.00

This problem set consists of 2 pages.

Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All items (item 1a), 1b), 2a), etc.) count 10 points except 4a) and 4b) which count 5 points each. If there is a problem you cannot solve, you may still use the result freely in later problems. At the end of the problem set you will find some information that might be useful along the way.

Problem 1. The function $f\colon [-\pi, \pi] \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 1 & \text{if } |x| < \frac{\pi}{2} \\ \frac{1}{2} & \text{if } |x| = \frac{\pi}{2} \\ 0 & \text{otherwise.} \end{cases}$$

a) (10 points) Show that the real Fourier series of f is

$$\frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos[(2n+1)x].$$

b) (10 points) Use the result in a) to find the sum of the alternating series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$. Explain your reasoning.

Problem 2. The map $F\colon C([0,1], \mathbb{R}) \to \mathbb{R}$ is given by $F(x) = x(0)x(1)$.

a) (10 points) Find the directional derivative $F'(x; r)$.

b) (10 points) Show that F is differentiable and find the derivative $F'(x)(r)$.

Problem 3. (10 points) Assume that (X, d) is a metric space and that $f\colon X \to \mathbb{R}$ is a continuous function. If $a \in \mathbb{R}$, we say that the equation $f(x) = a$ is *almost solvable* if there for each $\epsilon > 0$ is an $x \in X$ such that $|f(x) - a| < \epsilon$. Show that if X is compact and $f(x) = a$ is *almost solvable*, then there is an $x \in X$ such that $f(x) = a$. Find an example which shows that this is not always true for noncompact X.

(Continued on page 2.)

Problem 4. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space (i.e. a complete inner product space) over $\mathbb{R}$, and assume that $\{\mathbf{e}_n\}_{n\in\mathbb{N}}$ is an orthonormal set in H.

a) (5 points) Show that if $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ are sequences of real numbers such that $\sum_{n=1}^{\infty}\alpha_n\mathbf{e}_n = \sum_{n=1}^{\infty}\beta_n\mathbf{e}_n$, then $\alpha_n = \beta_n$ for all n.

Assume from now on that $\{\mathbf{e}_n\}_{n\in\mathbb{N}}$ is *not* a basis for H.

b) (5 points) Explain that there is an element $\mathbf{u} \in H$ such that $\mathbf{u} \neq \sum_{n=1}^{\infty}\alpha_n\mathbf{e}_n$, where $\alpha_n = \langle\mathbf{u}, \mathbf{e}_n\rangle$.

c) (10 points) Let $\mathbf{u}$ and α_n be as in b). Show that there is another element $\mathbf{v} \in H$ such that $\mathbf{v} = \sum_{n=1}^{\infty}\alpha_n\mathbf{e}_n$ and explain that $\mathbf{u}$ and $\mathbf{v}$ have the same "Fourier coefficients" with respect to $\{\mathbf{e}_n\}_{n\in\mathbb{N}}$ (i.e. $\langle\mathbf{u}, \mathbf{e}_n\rangle = \langle\mathbf{v}, \mathbf{e}_n\rangle$ for all $n \in \mathbb{N}$) even though $\mathbf{u} \neq \mathbf{v}$.

Problem 5. In this problem, X is the set of all continuous functions $f\colon [0,\infty) \to \mathbb{R}$ such that $\lim_{x\to\infty} f(x) = 0$.

a) (10 points) Show that all functions in X are bounded.

b) (10 points) Show that

$$\|f\| = \sup\{|f(x)| : x \in [0,\infty)\}$$

defines a norm on X. (You don't need to check that X is a vector space.)

c) (10 points) For $n \in \mathbb{N}$, let $e_n \in X$ be the function

$$e_n(x) = \begin{cases} 1 - |x-n| & \text{if } |x-n| < 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Make a sketch of e_n and find $\|e_n - e_m\|$ when $n \neq m$. Show that the unit sphere $B = \{f \in X : \|f\| = 1\}$ in X is not compact.

d) (10 points) Show that $(X, \|\cdot\|)$ is complete.

GOOD LUCK!

Some information you might find useful

Real Fourier coefficients:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad\qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Axioms for norms:

(i) $\|\mathbf{u}\| \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.

(ii) $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$.

(iii) $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

Basis: $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ is a *basis* for a normed space X if every $\mathbf{x} \in X$ can be written as a sum $\mathbf{x} = \sum_{n=1}^{\infty}\alpha_n\mathbf{u}_n$ in a unique way.

Definition of directional derivatives:

$$\mathbf{F}'(\mathbf{a};\mathbf{r}) = \lim_{t\to 0}\frac{\mathbf{F}(\mathbf{a}+t\mathbf{r}) - \mathbf{F}(\mathbf{a})}{t}.$$

THE END