

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT2400 — Real Analysis

Day of examination: Wednesday, May 29th, 2024

Examination hours: 15.00–19.00

This problem set consists of 8 pages.

Appendices: None

Permitted aids: No aids are allowed.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a

Consider the series

$$S(x) = \sum_{n=1}^{\infty} \frac{x}{n^2 + x}, \quad x \in [0, a].$$

for a fixed $a > 0$. Show that the series converges uniformly (on $[0, a]$) and find the derivative of the series with respect to x (justify your answer).

Answer: Observe that for all $x \in [0, a]$, each term of the series satisfies

$$\left| \frac{x}{n^2 + x} \right| \leq \frac{a}{n^2}.$$

The series $\sum_{n=1}^{\infty} \frac{a}{n^2}$ is convergent. Hence, by the Weierstrass M -test, the series $S(x)$ converges uniformly on $[0, a]$.

Differentiating term-by-term gives

$$\begin{aligned} S'(x) &= \sum_{n=1}^{\infty} \frac{d}{dx} \left(\frac{x}{n^2 + x} \right) \\ &= \sum_{n=1}^{\infty} \frac{n^2}{(n^2 + x)^2}, \end{aligned}$$

The uniform convergence of the derivative series can be shown using the Weierstrass M -test, as each term is bounded above by $\frac{1}{n^2}$ for all $x \geq 0$:

$$\left| \frac{n^2}{(n^2 + x)^2} \right| \leq \frac{1}{n^2}, \quad \forall x \geq 0,$$

(Continued on page 2.)

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Therefore, the series $S(x)$ converges uniformly on $[0, a]$, and its derivative $S'(x)$ is given by the uniformly convergent series $\sum_{n=1}^{\infty} \frac{n^2}{(n^2+x)^2}$.

b

Consider a nonempty set $F \subset \mathbb{R}$ which is closed and bounded with respect to the standard metric $d(x, y) = |x - y|$. Show that $\inf F \in F$. Also, present an example of a set F that is bounded but not closed, in which $\inf F \notin F$.

Answer: Set $i = \inf F$. Since F is non-empty and bounded, $i \in \mathbb{R}$. By definition of infimum, for each $n \in \mathbb{N}$, there exists $x_n \in F$ such that

$$i \leq x_n < i + \frac{1}{n},$$

so that $x_n \rightarrow i$ as $n \rightarrow \infty$. Since F is closed, it contains all its limit points; hence $i \in F$.

For the example, consider the set $F = (0, 1)$, which is a bounded open interval in $\mathbb{R}$. Then $\inf F = 0 \notin F$.

c

Let $f : X \rightarrow Y$ be a continuous function between two metric spaces (X, d_X) and (Y, d_Y) . Show that if $K \subseteq X$ is compact, then $f(K) \subset Y$ is compact.

Answer: Take a sequence $\{y_n\}$ in the set $f(K)$. Since f maps elements from K to $f(K)$, for each y_n in the sequence, there is a corresponding x_n in K such that $f(x_n) = y_n$. The compactness of K implies that from the sequence $\{x_n\}$, we can extract a converging subsequence $\{x_{n_k}\}$ that converges to a limit x in K . Set $y = f(x) \in f(K)$. By the continuity of f , the sequence $\{y_{n_k} = f(x_{n_k})\}$ converges to y . Thus, we have shown that any sequence in $f(K)$ has a converging subsequence that also lies in $f(K)$. This fact establishes the compactness of $f(K)$.

d

Use (b) and (c) to prove that if a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous over a compact subset K of $\mathbb{R}$ (with the standard metric), then f attains its minimum value within K .

Answer: By (c), $f(K)$ is compact in $\mathbb{R}$. As compact sets in $\mathbb{R}$ are closed and bounded, $f(K)$ is both. Thus, by (b) with $F = f(K)$,

$$\inf_{x \in K} f(x) = \inf F \in F.$$

(Continued on page 3.)

As $F = f(K)$, this means that there exists a point $x_0 \in K$ such that

$$\inf_{x \in K} f(x) = f(x_0).$$

e

Let (X, d) be a metric space and consider a subset A of X , equipped with the subset metric d_A . This metric is defined by setting $d_A(x, y) = d(x, y)$ for all $x, y \in A$.

(i) If a subset $C \subset A$ is closed in the metric space (X, d) , prove that C is also closed in (A, d_A) .

(ii) Provide an example to illustrate that a subset C of A , while being closed in (A, d_A) , may not necessarily be closed in the larger space (X, d) . Justify your answer.

Answer: (i) A set is closed in a metric space if every convergent sequence in the set converges to a limit that is also in the set. Consider an arbitrary sequence $\{x_n\}$ in C that converges in (A, d_A) to a limit x . Since d_A is the restriction of d to A , this sequence also converges to x in (X, d) . Since C is closed in (X, d) , the limit x belongs to C .

(ii) Let $X = \mathbb{R}$, with the standard metric $d(x, y) = |x - y|$. Define $A = (0, 2)$ (which is an open interval in $\mathbb{R}$). Let $C = [1, 2) \subset A$. Clearly, C is not closed in (X, d_X) . However, the set C is closed in (A, d_A) . Indeed, the complement $A \setminus C = (0, 1)$ is open in (A, d_A) .

Alternatively, consider a sequence $\{x_n\}$ in C converging to a limit x in (A, d_A) . Then x must belong to C . Note here that the only potential problem is that x is 2, but 2 is not an element of the space A and therefore cannot be a limit point in C within A .

Problem 2

a

Let $\{f_n\}$ be a sequence of functions in $C([0, 1])$, where $\|\cdot\|_\infty$ is the sup-norm. Suppose that there exists a constant $M > 0$ such that for all n , we have

(i) $\|f_n\|_\infty \leq M$,

(ii) f_n is differentiable, and

(iii) $\|f'_n\|_\infty \leq M$.

Prove that $\{f_n\}$ has a convergent subsequence.

Answer: According to the hard part of the Arzelà-Ascoli Theorem and (i), it is sufficient to verify that the sequence $\{f_n\}$ is equicontinuous. Take $\varepsilon > 0$ and fix n . By the Intermediate Value Theorem, for all $x, y \in [0, 1]$, there exists some c between x and y such that

$$f_n(x) - f_n(y) = f'_n(c)(x - y).$$

(Continued on page 4.)

Therefore,

$$|f_n(x) - f_n(y)| \leq M|x - y|.$$

Let $\delta = \varepsilon/M$. Then it follows that

$$|x - y| < \delta \implies |f_n(x) - f_n(y)| < M \frac{\varepsilon}{M} = \varepsilon.$$

This holds uniformly in n . Hence, the sequence $\{f_n\}$ is equicontinuous.

b

Consider the operator K defined by

$$K(g)(x) = \int_0^x g(y) dy, \quad x \in [0, 1].$$

Explain that K maps $C([0, 1])$ to $C([0, 1])$, where $C([0, 1])$ is equipped with the sup (uniform) norm $\|\cdot\|_\infty$. Provide the definition of the operator norm $\|K\|$. Prove that

$$\|K\| \leq 1.$$

Answer: Note first that $f = K(g) \in C([0, 1])$ for any $g \in C([0, 1])$. Indeed, f is continuously differentiable since $f' = g$. This implies that f is continuous, and thus $K : C([0, 1]) \rightarrow C([0, 1])$.

The operator norm of K is defined as

$$\|K\| = \sup_{g \neq 0} \frac{\|K(g)\|_\infty}{\|g\|_\infty} = \sup_{\|g\|_\infty=1} \|K(g)\|_\infty.$$

For a given g with $\|g\|_\infty = 1$, we have

$$|K(g)(x)| = \left| \int_0^x g(y) dy \right| \leq \int_0^x |g(y)| dy \leq \int_0^1 1 dy = 1, \quad x \in [0, 1].$$

Therefore, $\|K(g)\|_\infty \leq 1$, and thus $\|K\| \leq 1$.

c

Let K be the operator from (b). We say that K is *compact* if for every bounded sequence $\{g_n\}$ in $C([0, 1])$, the sequence $\{K(g_n)\}$ has a convergent subsequence in $C([0, 1])$. Prove that K is a compact operator.

Answer: To prove that K is compact, we use problem (a) with $f_n = K(g_n)$, where $\{g_n\}$ is a bounded sequence in $C([0, 1])$, which means that there exists a constant $M > 0$ such that

$$\|g_n\|_\infty \leq M, \quad \forall n.$$

Firstly, considering the definition of the norm, and given that $\|K\| \leq 1$,

$$\|f_n\|_\infty = \|K(g_n)\|_\infty \leq \|K\| \|g_n\|_\infty \leq M,$$

(Continued on page 5.)

which establishes (i). Clearly, f_n is differentiable ($f'_n = g_n$), so that (ii) holds. Finally,

$$\|f'_n\|_\infty = \|g_n\|_\infty \leq M,$$

so that also (iii) holds. Thus, K is a compact operator.

Problem 3

a

Apply the Neumann Series theorem to prove that the following integral equation has a unique solution $f \in C([0, 1])$:

$$f(x) = u(x) + \int_0^1 \frac{1}{2} e^{-|x-y|} f(y) dy, \quad \text{for all } x \in [0, 1], \quad (1)$$

where the “input” $u \in C([0, 1])$ is a given function. As before, $C([0, 1])$ is equipped with the supremum norm $\|\cdot\|_\infty$. Moreover, show that this solution satisfies the following estimate:

$$\|f\|_\infty \leq 2\|u\|_\infty.$$

Hint: Consider the inverse of $(I - T)$, where T is a suitably defined bounded linear operator that maps from $C([0, 1])$ to $C([0, 1])$, where these properties must be verified. When checking that $T(f)$ is a continuous function, you may want to use that $|e^{-|x_1|} - e^{-|x_2|}| \leq |x_1 - x_2|$.

Answer: Set $K(x, y) = \frac{1}{2} e^{-|x-y|}$. Then $\|K\|_\infty = \frac{1}{2}$. Introducing the operator

$$T(f)(x) = \int_0^1 K(x, y) f(y) dy,$$

we may write the integral equation as

$$f = u + T(f). \quad (2)$$

Based on the continuity of K , it is clear that $T(f) = T(f)(x)$ is a continuous function on $[0, 1]$:

$$|T(f)(x_1) - T(f)(x_2)| \leq \|f\|_\infty \int_0^1 |K(x_1, y) - K(x_2, y)| dy.$$

Using the hint, we arrive at

$$|T(f)(x_1) - T(f)(x_2)| \leq \frac{1}{2} \|f\|_\infty |x_1 - x_2|, \quad \forall x_1, x_2 \in [0, 1],$$

which implies that $T(f)$ is continuous. Hence,

$$T : C([0, 1]) \rightarrow C([0, 1]).$$

Let us verify that T is a bounded linear operator. Linearity is checked as

(Continued on page 6.)

follows: for $f_1, f_2 \in C([0, 1])$ and $a_1, a_2 \in \mathbb{R}$, we have

$$\begin{aligned} T(a_1 f_1 + a_2 f_2)(x) &= \int_0^1 K(x, y)(a_1 f_1(y) + a_2 f_2(y)) dy \\ &= a_1 \int_0^1 K(x, y) f_1(y) dy + a_2 \int_0^1 K(x, y) f_2(y) dy \\ &= a_1 T(f_1)(x) + a_2 T(f_2)(x). \end{aligned}$$

To verify that T is bounded, recall that

$$\|T\| = \sup_{\|f\|_\infty=1} \|T(f)\|_\infty.$$

For $f \in C([0, 1])$ with $\|f\|_\infty = 1$, and any $x \in [0, 1]$,

$$\begin{aligned} |T(f)(x)| &\leq \left| \int_0^1 K(x, y) f(y) dy \right| \\ &\leq \|K\|_\infty \|f\|_\infty = \frac{1}{2}. \end{aligned}$$

Therefore,

$$\|T\| \leq \frac{1}{2}.$$

Let us consider that $(C([0, 1]), \|\cdot\|_\infty)$ is a complete normed space. The mapping

$$T : C([0, 1]) \rightarrow C([0, 1])$$

is a bounded, linear operator where its norm satisfies

$$\|T\| \leq \frac{1}{2} < 1. \quad (3)$$

Based on the Neumann Series theorem, the operator $I - T$ is invertible, and its inverse, denoted as $(I - T)^{-1}$, has a norm bound given by $\|(I - T)^{-1}\| \leq 1/(1 - \|T\|)$. Consequently, we can uniquely determine f from the equation $f = u + T(f)$ as

$$f = (I - T)^{-1}(u).$$

Furthermore,

$$\|f\|_\infty = \|(I - T)^{-1}(u)\|_\infty \leq \|(I - T)^{-1}\| \|u\|_\infty,$$

with the bound for $\|(I - T)^{-1}\|$ being

$$\|(I - T)^{-1}\| \leq \frac{1}{1 - \|T\|} \leq \frac{1}{1 - 1/2} = 2.$$

In summary,

$$\|f\|_\infty \leq 2\|u\|_\infty.$$

b

A sequence of continuous functions on the interval $[0, 1]$, denoted by $\{f_n\}$, is defined inductively. Start with an arbitrary function $f_0 \in C([0, 1])$. For each $n = 1, 2, \dots$, define the function f_{n+1} in terms of the previous function f_n by the relation

$$f_{n+1}(x) = u(x) + \int_0^1 K(x, y) f_n(y) dy,$$

where $u \in C([0, 1])$ is given and $K(x, y) = \frac{1}{2}e^{-|x-y|}$.

Use the Neumann Series theorem to prove that this sequence $\{f_n\}$ converges uniformly to the unique solution $f \in C([0, 1])$ of the integral equation (1).

Answer: Introducing the operator

$$T(f)(x) = \int_0^1 K(x, y) f(y) dy,$$

we may write the recursive equation as

$$f_{n+1} = u + T(f_n). \quad (4)$$

Observe that

$$\begin{aligned} f_1 &= u + T(f_0), \\ f_2 &= u + T(f_1) = u + T(u + T(f_0)) \\ &= u + T(u) + T^2(f_0). \end{aligned}$$

More generally, for $n = 1, 2, \dots$, we have

$$f_n = \sum_{k=0}^{n-1} T^k(u) + T^n(f_0),$$

where, by (3),

$$\|T^n(f_0)\|_\infty \leq \|T\|^n \|f_0\|_\infty \leq \left(\frac{1}{2}\right)^n \|f_0\|_\infty \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore, by the Neumann Series theorem, it follows that

$$\lim_{n \rightarrow \infty} f_n = \sum_{i=0}^{\infty} T^i(u) = (I - T)^{-1}(u) = f,$$

where f is the unique solution of $f = u + T(f) \iff f = (I - T)^{-1}(u)$. More precisely, since the series converges in operator norm,

$$\|f_n - f\|_\infty \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

(Continued on page 8.)

Some useful information

Proposition (hard part of the Arzelà-Ascoli Theorem). Assume that (X, d) is a compact metric space, and let $\{f_n\}$ be a bounded and equicontinuous sequence in $C(X, \mathbb{R})$. Then $\{f_n\}$ has a subsequence converging in $C(X, \mathbb{R})$.

Theorem (Neumann Series). Assume that U is a complete, normed space and that $A : U \rightarrow U$ is a bounded, linear operator with $\|A\| < 1$. Then $I - A$ is invertible and

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n,$$

where the series converges in operator norm. Moreover,

$$\|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

THE END