

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT2400 — Real analysis

Day of examination: 12 August 2026

Examination hours: 9:00 – 13:00

This problem set consists of 6 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note:

- The exam consists of 10 problems/subproblems, and each of these counts 10 points, for a maximum of 100 points.
- You will find a list of definitions and properties that you might (or might not) find useful on page 6.
- You may write your answers in either English or Norwegian.

Problem 1. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \rightarrow Y$ be continuous. Show that $f^{-1}(E)$ is closed for every closed $E \subseteq Y$. Is the same true if “closed” is replaced by “compact”?

Solution: We use the following definition of continuity: For every sequence $(x_n)_{n \in \mathbb{N}}$ in X converging to some $x \in X$, we have

$$\lim_{n \rightarrow \infty} f(x_n) = f(x).$$

Let $E \subseteq Y$ be closed, and let $(x_n)_{n \in \mathbb{N}}$ be any sequence in $f^{-1}(E)$ converging to some $x \in X$. We wish to show that $x \in f^{-1}(E)$. Let $y_n := f(x_n)$, which all lie in E , since $x_n \in f^{-1}(E)$. By continuity of f , the sequence $(y_n)_{n \in \mathbb{N}}$ converges to $y := f(x)$. Since E is closed, we get $y \in E$. And since $f(x) = y$, we can conclude that $x \in f^{-1}(E)$.

The same is not true if “closed” is replaced by “compact”. For instance, let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the continuous function $f(t) := \cos(t)$. The set $E := [-2, 2]$ is compact in $\mathbb{R}$, but $f^{-1}(E) = \mathbb{R}$ is not compact since it is unbounded.

Problem 2.

2(a) Let (X, d_X) and (Y, d_Y) be metric spaces. Let $D \subseteq X$ be a dense subset, and let $f, g: X \rightarrow Y$ be continuous functions such that $f(x) = g(x)$ for all $x \in D$. Prove that $f = g$.

(Continued on page 2.)

2(b) Let (X, d_X) be a metric space. Let $D \subseteq X$ be a subset with the following property:

if $f, g: X \rightarrow \mathbb{R}$ are continuous functions such that $f(x) = g(x)$ for all $x \in D$, then $f = g$.

Prove that D is dense in X .

2(c) Let $V := C([0, 3], \mathbb{R})$ be the space of continuous functions $f: [0, 3] \rightarrow \mathbb{R}$, equipped with the supremum norm $\|f\|_{\sup} := \sup_{t \in [0, 3]} |f(t)|$. Prove that $A: V \rightarrow \mathbb{R}$ defined by $A(f) := \int_0^3 f(t) dt$ is a bounded linear operator, and compute its operator norm.

2(d) Let V be as in the previous problem. Let $B: V \rightarrow \mathbb{R}$ be another bounded linear operator with the property that

$$B(p_k) = \frac{3^{k+1}}{k+1} \quad \text{for any } k = 0, 1, 2, \dots, \text{ where } p_k(t) := t^k \text{ for } t \in [0, 3].$$

Prove that $B = A$.

Hint: You might need to apply Weierstrass' approximation theorem.

Solution:

2(a) Since D is dense in X , there is for every $x \in X$ a sequence $(x_n)_{n \in \mathbb{N}}$ in D such that $x_n \rightarrow x$. By continuity of f and g , we have $f(x) = \lim_n f(x_n)$ and $g(x) = \lim_n g(x_n)$. Since f and g agree on every x_n , we conclude that $f(x) = g(x)$.

2(b) Assume conversely that D is not dense in X . Then there is some $y \in X$ and $r > 0$ such that $B(y; r) \subsetneq D$. Define

$$f(x) := \begin{cases} r - d(x, y) & \text{if } x \in B(y; r) \\ 0 & \text{else,} \end{cases}$$

and let $g \equiv 0$. Then f and g are continuous, and $f(x) = g(x) = 0$ on D , but $f \neq g$ — a contradiction. Hence, D must be dense in X .

2(c) The function A is clearly linear:

$$A(\alpha f + \beta g) = \int_0^3 \alpha f + \beta g dt = \alpha \int_0^3 f dt + \beta \int_0^3 g dt = \alpha A(f) + \beta A(g).$$

We have

$$|A(f)| \leq \int_0^3 |f(t)| dt \leq \int_0^3 \|f\|_{\sup} dt = 3\|f\|_{\sup},$$

so A is bounded with operator norm at most 3. If $f \equiv 1$ then

$$|A(f)| = \int_0^3 1 dt = 3 = 3\|f\|_{\sup},$$

so in fact $\|A\|_{\mathcal{L}} = 3$.

(Continued on page 3.)

2(d) The property stated in the problem can be written as $B(p_k) = \int_0^3 p_k(t) dt = A(p_k)$ for every $k \in \mathbb{N}_0$. By linearity, we get $B(p) = A(p)$ for every polynomial p . By Weierstrass' approximation theorem, the set $D := \{\text{all polynomials } p \in V\}$ is dense in V . We know that all bounded linear operators on V are continuous, so by Problem (a), we can conclude that $B = A$.

Problem 3. Let $(X, \mathcal{A})$ be a measurable space. A *probability measure* on $(X, \mathcal{A})$ is a measure μ satisfying $\mu(X) = 1$. Show that if $\mu_1, \mu_2, \dots$ are probability measures, then so is λ , where

$$\lambda(E) := \sum_{n=1}^{\infty} 2^{-n} \mu_n(E) \quad \text{for } E \in \mathcal{A}.$$

Solution: First of all, λ is well-defined on $\mathcal{A}$, by Weierstrass' M-test: Since $\mu_n(E) \in [0, 1]$ for every $n \in \mathbb{N}$, we get

$$\sum_{n=1}^{\infty} |2^{-n} \mu_n(E)| \leq \sum_{n=1}^{\infty} 2^{-n} = 1.$$

Next, λ is a measure:

- Clearly, $\lambda(E) \geq 0$ for every $E \in \mathcal{A}$.
- $\lambda(\emptyset) = \sum_n 2^{-n} \mu_n(\emptyset) = 0$.
- If $E_1, E_2, \dots \in \mathcal{A}$ are disjoint then

$$\begin{aligned} \lambda\left(\bigcup_{k \in \mathbb{N}} E_k\right) &= \sum_{n=1}^{\infty} 2^{-n} \mu_n\left(\bigcup_{k \in \mathbb{N}} E_k\right) = \sum_{n=1}^{\infty} 2^{-n} \sum_{k \in \mathbb{N}} \mu_n(E_k) \\ &= \sum_{k \in \mathbb{N}} \sum_{n=1}^{\infty} 2^{-n} \mu_n(E_k) = \sum_{k \in \mathbb{N}} \lambda(E_k), \end{aligned}$$

where we in the third step could interchange the series since all series involved are convergent.

Finally, we have

$$\lambda(X) = \sum_{n=1}^{\infty} 2^{-n} \mu_n(X) = \sum_{n=1}^{\infty} 2^{-n} = 1.$$

Problem 4. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f: X \rightarrow \overline{\mathbb{R}}$ be an integrable, extended real-valued function. Prove that f is finite almost everywhere.

Solution: Let $E := \{x : f(x) \text{ is not finite}\} = |f|^{-1}(\{\infty\})$. If $\varphi(x) := \infty \mathbb{1}_E(x)$ then $|f|(x) \geq \varphi(x)$ for every $x \in X$. We have

$$\infty \cdot \mu(E) = \int \varphi d\mu \leq \int |f| d\mu < \infty,$$

(Continued on page 4.)

so necessarily $\mu(E) = 0$. Hence, f is finite almost everywhere.

Problem 5. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f_n: X \rightarrow [0, \infty)$ be a sequence of integrable functions converging pointwise to an integrable function $f: X \rightarrow [0, \infty)$. Assume in addition that

$$\int f_n d\mu \rightarrow \int f d\mu \quad \text{as } n \rightarrow \infty.$$

Use Fatou's lemma to prove that also $\int_E f_n d\mu \rightarrow \int_E f d\mu$ as $n \rightarrow \infty$ for every measurable $E \subseteq X$.

Solution: Let $E \in \mathcal{A}$. Applying Fatou's lemma to $f \mathbb{1}_E = \lim_n f_n \mathbb{1}_E$, we get

$$\int_E f d\mu \leq \liminf_{n \rightarrow \infty} \int_E f_n d\mu. \quad (*)$$

Applying Fatou's lemma to $f \mathbb{1}_{E^c}$ yields

$$\begin{aligned} \int_{E^c} f d\mu &= \int \liminf_{n \rightarrow \infty} f_n \mathbb{1}_{E^c} d\mu \leq \liminf_{n \rightarrow \infty} \int f_n \mathbb{1}_{E^c} d\mu \\ &= \liminf_{n \rightarrow \infty} \left(\int f_n d\mu - \int_E f_n d\mu \right) \\ &= \int f d\mu + \liminf_{n \rightarrow \infty} \left(- \int_E f_n d\mu \right), \end{aligned}$$

where we have used the fact that $\int f_n d\mu \rightarrow \int f d\mu$. Since $\liminf(-\dots) = -\limsup \dots$, we get after reorganizing

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_E f_n d\mu &\leq \int f d\mu - \int_{E^c} f d\mu = \int_E f d\mu \\ &= \int f \mathbb{1}_E d\mu \leq \liminf_{n \rightarrow \infty} \int f_n \mathbb{1}_E d\mu \\ &= \liminf_{n \rightarrow \infty} \int_E f_n d\mu, \end{aligned}$$

where we in the third step have used (*). The only way that a “lim sup” can be upper bounded by a “lim inf” is that there is equality between them, and that the sequence converges. Hence, every inequality above must be an equality, whence

$$\lim_{n \rightarrow \infty} \int_E f_n d\mu = \int_E f d\mu.$$

Problem 6. Let $F: \ell^2(\mathbb{R}) \rightarrow \ell^1(\mathbb{R})$ be given by

$$F(a) := (a_n a_{2n})_{n \in \mathbb{N}} \quad \text{for } a = (a_1, a_2, \dots) \in \ell^2(\mathbb{R}).$$

6(a) Show that F is well-defined, that is, that the above formula defines a function from $\ell^2(\mathbb{R})$ to $\ell^1(\mathbb{R})$.

6(b) Prove that F is differentiable, and compute its derivative (also called *Fréchet derivative*).

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Solution:

6(a) To check that F is well-defined, we need to check that $F(a) \in \ell^1(\mathbb{R})$ for every $a \in \ell^2(\mathbb{R})$. Using Hölder's inequality we get

$$\begin{aligned}\|F(a)\|_{\ell^1} &= \sum_{n \in \mathbb{N}} |a_n a_{2n}| \leq \left(\sum_{n \in \mathbb{N}} a_n^2 \right)^{1/2} \left(\sum_{n \in \mathbb{N}} a_{2n}^2 \right)^{1/2} \\ &\leq \|a\|_{\ell^2}^2 < \infty.\end{aligned}$$

Hence, $F(a)$ indeed lies in $\ell^1(\mathbb{R})$ for every $a \in \ell^2(\mathbb{R})$, so F is well-defined.

6(b) To find its derivative, we first find its directional derivative. For any $a, b \in \ell^2(\mathbb{R})$ and $h \in \mathbb{R}$ we get

$$\begin{aligned}F(a + hb) - F(a) &= ((a_n + hb_n)(a_{2n} + hb_{2n}))_{n \in \mathbb{N}} - (a_n a_{2n})_{n \in \mathbb{N}} \\ &= (h(a_n b_{2n} + a_{2n} b_n))_{n \in \mathbb{N}} + h^2(b_n b_{2n})_{n \in \mathbb{N}},\end{aligned}$$

and we note that both of the above vectors lie in $\ell^1(\mathbb{R})$, for the same reason that $F(a) \in \ell^1(\mathbb{R})$. The directional derivative at a in direction b is therefore

$$\begin{aligned}F'(a; b) &= \lim_{h \rightarrow 0} \frac{1}{h} (F(a + hb) - F(a)) \\ &= \lim_{h \rightarrow 0} ((a_n b_{2n} + a_{2n} b_n)_{n \in \mathbb{N}} + h(b_n b_{2n})_{n \in \mathbb{N}}) \\ &= (a_n b_{2n} + a_{2n} b_n)_{n \in \mathbb{N}}.\end{aligned}$$

If F is (Fréchet) differentiable at a , then $F'(a)(r) = F'(a; r)$ for every $r \in \ell^2$. We check that the latter defines a bounded linear operator on ℓ^2 . Clearly, $r \mapsto F'(a; r)$ is linear. Next,

$$\|F'(a; r)\|_{\ell^1} \leq \sum_{n \in \mathbb{N}} |a_n b_{2n}| + |a_{2n} b_n| \leq \|a\|_{\ell^2} \|b\|_{\ell^2},$$

the last step following from the same computation as previously. Hence, $\|F'(a; \cdot)\|_{\mathcal{L}} \leq \|a\|_{\ell^2}$, so it is bounded.

By the same computation as above we get

$$\|F(a + r) - F(a) - F'(a; r)\|_{\ell^1} = \|(r_n r_{2n})_{n \in \mathbb{N}}\|_{\ell^1} \leq \|r\|_{\ell^2}^2.$$

Hence, $(F(a + r) - F(a) - F'(a; r))/\|r\|_{\ell^2} \rightarrow 0$ as $r \rightarrow 0$, so F is indeed differentiable, with derivative

$$F'(a)(r) = F'(a; r) = (a_n r_{2n} + a_{2n} r_n)_{n \in \mathbb{N}}.$$

Some information you might find useful

Limit inferior and superior

If $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are sequences of real numbers such that $(a_n)_{n \in \mathbb{N}}$ converges, then

$$\liminf_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \liminf_{n \rightarrow \infty} b_n$$

(and likewise for “lim sup”). For any sequence $(b_n)_{n \in \mathbb{N}}$ we have

$$\liminf_{n \rightarrow \infty} (-b_n) = -\limsup_{n \rightarrow \infty} b_n.$$

The vector spaces $\ell^p(\mathbb{R})$:

For $p \in [1, \infty)$, the ℓ^p spaces are defined by

$$\ell^p(\mathbb{R}) := \{a = (\alpha_1, \alpha_2, \dots) : \alpha_1, \alpha_2, \dots \in \mathbb{R}, \|a\|_{\ell^p} < \infty\}$$

where $\|\cdot\|_{\ell^p}$ is the ℓ^p norm,

$$\|a\|_{\ell^p} := \left(\sum_{k \in \mathbb{N}} |\alpha_k|^p \right)^{1/p} \quad \text{for } a = (\alpha_1, \alpha_2, \dots).$$

Hölder's inequality/the Cauchy–Schwarz inequality:

For any $a, b \in \ell^2(\mathbb{R})$, we have

$$\sum_{k \in \mathbb{N}} |\alpha_k \beta_k| \leq \|a\|_{\ell^2} \|b\|_{\ell^2}$$

where $a = (\alpha_1, \alpha_2, \dots)$, $b = (\beta_1, \beta_2, \dots)$.

GOOD LUCK!