

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT2400 — Real analysis

Day of examination: 3 June 2026

Examination hours: 15:00–19:00

This problem set consists of 9 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note:

- The exam consists of 10 problems/subproblems, and each of these counts 10 points, for a maximum of 100 points.
- You will find a list of definitions and properties that you might (or might not) find useful on page 9.
- You may write your answers in either English or Norwegian.

Problem 1. Let (X, d_X) and (Y, d_Y) be metric spaces, $E \subseteq X$ an open set, and $f_1, f_2: X \rightarrow Y$ two continuous functions. Assume that

$$f_1(x) = f_2(x) \quad \forall x \in E.$$

Show that also $f_1(x) = f_2(x)$ for all $x \in \overline{E}$ (the closure of E).

Solution: We have $\overline{E} = E \cup \partial E$, so it remains to show that $f_1(x) = f_2(x)$ for all $x \in \partial E$. Let $x \in \partial E$, let $\varepsilon > 0$, and let $\delta_1, \delta_2 > 0$ be such that $d_Y(f_k(x), f_k(y)) < \varepsilon$ when $d_X(x, y) < \delta_k$ (for $k = 1, 2$). Since $x \in \partial E$, there is some $y \in E$ such that $d_X(x, y) < \min(\delta_1, \delta_2)$. Then $f_1(y) = f_2(y)$. Hence,

$$\begin{aligned} d_Y(f_1(x), f_2(x)) &\leq d_Y(f_1(x), f_1(y)) + d_Y(f_1(y), f_2(y)) + d_Y(f_2(y), f_2(x)) \\ &\leq \varepsilon + 0 + \varepsilon = 2\varepsilon. \end{aligned}$$

Since this holds for any $\varepsilon > 0$, we get $d_Y(f_1(x), f_2(x)) = 0$, whence $f_1(x) = f_2(x)$.

Alternative solution: Define $g: X \rightarrow Y$ by $g(x) := d_Y(f_1(x), f_2(x))$. Then g is continuous and $g(x) = 0$ for all $x \in E$. The same technique as above shows that $g(x) = 0$ also for $x \in \partial E$.

Alternative solution: We have $\overline{E} = E \cup \partial E$, so it remains to show that $f_1(x) = f_2(x)$ for all $x \in \partial E$. Let $x \in \partial E$. Then $f_k(y) \rightarrow f_k(x)$ as $y \rightarrow x$, by continuity of f_k (for $k = 1, 2$). Since $x \in \partial E$, there are

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$y_1, y_2, \dots \in E$ such that $y_n \rightarrow x$ as $n \rightarrow \infty$. Since $f_1(y_n) = f_2(y_n)$ for all n , we get

$$f_1(x) = \lim_{n \rightarrow \infty} f_1(y_n) = \lim_{n \rightarrow \infty} f_2(y_n) = f_2(x).$$

Problem 2. Let $\mathcal{A}$ be a σ -algebra on $\mathbb{R}$, and define the *Dirac measure*

$$\delta(E) := \begin{cases} 1 & \text{if } 0 \in E \\ 0 & \text{if } 0 \notin E \end{cases} \quad \text{for } E \in \mathcal{A}.$$

2(a) Show that δ is a measure on $(\mathbb{R}, \mathcal{A})$.

2(b) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be measurable with respect to $\mathcal{A}$. Show that

$$\int f d\delta = f(0).$$

Solution:

2(a) We clearly have $\delta(\emptyset) = 0$. If $E_1, E_2, \dots \in \mathcal{A}$ are disjoint, we have

- if $0 \in E_n$ for some $n \in \mathbb{N}$, then $0 \in \cup_{k \in \mathbb{N}} E_k$, so

$$\delta\left(\bigcup_{k \in \mathbb{N}} E_k\right) = 1,$$

while

$$\sum_{k \in \mathbb{N}} \delta(E_k) = \underbrace{\delta(E_n)}_{=1} + \sum_{k \neq n} \underbrace{\delta(E_k)}_{=0} = 1,$$

since $0 \notin E_k$ for all $k \neq n$.

- if $0 \notin E_k$ for all $k \in \mathbb{N}$, then also $0 \notin \cup_{k \in \mathbb{N}} E_k$, so

$$\delta\left(\bigcup_{k \in \mathbb{N}} E_k\right) = 0 = \sum_{k \in \mathbb{N}} \delta(E_k).$$

Hence, δ is σ -additive, so it is a measure.

2(b) We first show the result for simple functions, then approximate an arbitrary function by measurable functions. Let $\varphi = \sum_{k=1}^N \alpha_k \mathbb{1}_{E_k}$ be any simple function. Assuming that $E_1, \dots, E_N$ are disjoint and that $E_1 \cup \dots \cup E_N = X$, there is exactly one set E_n such that $0 \in E_n$. Then

$$\int \varphi d\delta = \sum_{k=1}^N \alpha_k \delta(E_k) = \alpha_n = \varphi(0).$$

For an arbitrary nonnegative measurable function $f: X \rightarrow \mathbb{R}_+$, let $\varphi_1, \varphi_2, \dots$ be an increasing sequence of nonnegative simple functions

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converging pointwise to f . Then, by definition of the integral of f ,

$$\int f \, d\delta = \lim_{k \rightarrow \infty} \int \varphi_k \, d\delta = \lim_{k \rightarrow \infty} \varphi_k(0) = f(0).$$

Finally, for arbitrary measurable $f: X \rightarrow \mathbb{R}$, decompose f into its positive and negative part, $f = f_+ - f_-$. Then

$$\int f \, d\delta = \int f_+ \, d\delta - \int f_- \, d\delta = f_+(0) - f_-(0) = f(0).$$

Problem 3. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f_1, f_2, \dots: X \rightarrow \mathbb{R}$ be measurable functions such that $|f_k| \leq g$ for some integrable function $g: X \rightarrow [0, \infty)$, and such that $(f_k)_{k \in \mathbb{N}}$ converges pointwise to some function $f: X \rightarrow \mathbb{R}$. Show that f and $f_1, f_2, \dots$ are integrable, and that $f_k \rightarrow f$ in the L^1 norm as $k \rightarrow \infty$.

Solution: First of all, f is measurable, since it is the pointwise limit of measurable functions. Since $|f_k| \leq g$, we have $\int |f_k| \, d\mu \leq \int g \, d\mu < \infty$, so f_k is integrable, and since $f_k \rightarrow f$ pointwise, also $|f| \leq g$, whence $\int |f| \, d\mu \leq \int g \, d\mu < \infty$, so f is also integrable.

We wish to show that $\int |f_k - f| \, d\mu \rightarrow 0$ as $k \rightarrow \infty$, that is, $\int h_k \, d\mu \rightarrow 0$ as $k \rightarrow \infty$ where $h_k := |f_k - f|$. We have $0 \leq h_k \leq |f_k| + |f| \leq 2g$, so the sequence $(h_k)_{k \in \mathbb{N}}$ is dominated by the integrable function $2g$. Moreover, the sequence converges pointwise to 0, so we can invoke the Lebesgue Dominated Convergence Theorem to conclude that

$$\lim_{k \rightarrow \infty} \|f_k - f\|_{L^1} = \lim_{k \rightarrow \infty} \int h_k \, d\mu = \int \lim_{k \rightarrow \infty} h_k \, d\mu = \int 0 \, d\mu = 0.$$

Problem 4. For an interval $[a, b] \subseteq \mathbb{R}$ and some $N \in \mathbb{N}$, define the set of polynomials on $[a, b]$ of order up to N as

$$\mathcal{P}_N([a, b]) := \left\{ p: [a, b] \rightarrow \mathbb{R} : p(t) = \sum_{k=0}^N \alpha_k t^k \text{ for } \alpha_0, \dots, \alpha_N \in \mathbb{R} \right\},$$

and for some $M > 0$, define the subset

$$A := \left\{ p \in \mathcal{P}_N([a, b]) : p(t) = \sum_{k=0}^N \alpha_k t^k \text{ for } \alpha_0, \dots, \alpha_N \in [-M, M] \right\}.$$

Show that for any sequence $p_1, p_2, \dots \in A$, there is a subsequence $p_{n_1}, p_{n_2}, \dots$ converging uniformly to some continuous function p .

Solution: Let $p \in A$, so there are $\alpha_0, \dots, \alpha_N \in [-M, M]$ such that $p(t) = \sum_{k=0}^N \alpha_k t^k$ for all $t \in [a, b]$. Let $c := \max(|a|, |b|)$. Then

$$|p(t)| \leq \sum_{k=0}^N |\alpha_k| |t|^k \leq \sum_{k=0}^N M c^k =: C_1,$$

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which is bounded independently of p . Moreover,

$$|p'(t)| = \left| \sum_{k=1}^N \alpha_k k t^{k-1} \right| \leq \sum_{k=1}^N |\alpha_k| k |t|^{k-1} \leq \sum_{k=1}^N M k |c|^{k-1} =: C_2,$$

which is also bounded independently of p . It follows that any sequence $p_1, p_2, \dots \in A$ is uniformly bounded (by C_1), and has uniformly bounded first derivative (with upper bound C_2). As a consequence, the functions are Lipschitz continuous with Lipschitz constant at most C_2 — in particular, the sequence is equicontinuous. Since $[a, b]$ is compact, it follows from Arzela–Ascoli that $(p_n)_{n \in \mathbb{N}}$ has a uniformly convergent subsequence, converging to some continuous function p . (This function will also lie in A , but you are not required to show that.)

Alternative solution: Write $p_n(t) = \sum_{k=0}^N \alpha_{n,k} t^k$ for $n \in \mathbb{N}$, and let $x_n := (\alpha_{n,0}, \dots, \alpha_{n,N}) \in \mathbb{R}^{N+1}$. Then the sequence $(x_n)_{n \in \mathbb{N}}$ lies in the closed and bounded set $[-M, M]^{N+1}$, so by the Bolzano–Weierstrass theorem, there is a subsequence $(x_{n(m)})_{m \in \mathbb{N}}$ converging to some $x \in \mathbb{R}^{N+1}$. If $x = (\beta_0, \dots, \beta_N)$ are the components of x , and $p(t) := \sum_{k=0}^N \beta_k t^k$, then

$$\begin{aligned} |p_{n(m)}(t) - p(t)| &= \left| \sum_{k=0}^N (\alpha_{n(m),k} - \beta_k) t^k \right| \leq \sum_{k=0}^N |\alpha_{n(m),k} - \beta_k| |t|^k \\ &\leq \|x_{n(m)} - x\| \sum_{k=0}^n |t|^k \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$, uniformly in $t \in [a, b]$. Hence, $p_{n(m)} \rightarrow p$ uniformly.

Problem 5. Let $X := \ell^2(\mathbb{R})$, equipped with the ℓ^2 norm, and let $Y := C([0, 2\pi], \mathbb{R})$, equipped with the supremum norm. Define $F: X \rightarrow Y$ by

$$F(a) := \left(t \mapsto \sum_{k \in \mathbb{N}} a_k^2 \cos(kt) \right) \quad \text{for } a = (a_1, a_2, \dots) \in X. \quad (1)$$

5(a) Show that F is well-defined. That is, show that (1) indeed defines a function F from X to Y .

5(b) Show that F has a directional derivative $F'(a; r)$ (also called *Gateaux derivative*) for any $a, r \in X$, and compute this derivative.

5(c) Show that F is differentiable (also called *Fréchet differentiable*) at any $a \in X$.

Solution:

5(a) Fix an $a \in X$. We use Weierstrass' M-test. Since each term in the series (1) can be uniformly bounded as

$$|a_k^2 \cos(kt)| \leq a_k^2,$$

and since $\sum_{k \in \mathbb{N}} a_k^2 < \infty$, Weierstrass' M-test guarantees that the series

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in (1) converges uniformly for any fixed $a \in X$. Since each term in the series is a continuous function, and the series converges uniformly, the limit will also be continuous, and therefore lies in Y . Thus, $F(a)$ is an element of Y for all $a \in X$, so F is well-defined.

Alternative solution: Fix $a \in X$, and define the partial sums $S_n(t) := \sum_{k=1}^n a_k^2 \cos(kt)$. Being a sum of continuous functions, S_n is continuous, so $S_n \in Y$ for all $n \in \mathbb{N}$. We have for any $n, m \in \mathbb{N}$ (assuming, say, $m \leq n$)

$$|S_n - S_m| = \left| \sum_{k=m+1}^n a_k^2 \cos(kt) \right| \leq \sum_{k=m+1}^n |a_k|^2 \underbrace{|\cos(kt)|}_{\leq 1} \leq \sum_{k=m+1}^{\infty} |a_k|^2.$$

Since the series $\sum_{k=1}^{\infty} |a_k|^2$ converges (by the assumption $a \in X$), the above goes to zero as $m, n \rightarrow \infty$. This shows that $(S_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. Since Y is complete, the series converges to some $S \in Y$. In particular, $F(a) = S$. This proves that F is well-defined at any $a \in X$.

5(b) The directional derivative is defined as $F'(a; r) := \lim_{h \rightarrow 0} (F(a + hr) - F(a))/h$. We have for any $t \in [0, 2\pi]$

$$\begin{aligned} \frac{F(a + hr) - F(a)}{h}(t) &= \frac{1}{h} \left(\sum_{k \in \mathbb{N}} (a_k + hr_k)^2 \cos(kt) - \sum_{k \in \mathbb{N}} a_k^2 \cos(kt) \right) \\ &= \frac{1}{h} \sum_{k \in \mathbb{N}} (2ha_k r_k + h^2 r_k^2) \cos(kt) \\ &= 2 \sum_{k \in \mathbb{N}} a_k r_k \cos(kt) + h \sum_{k \in \mathbb{N}} r_k^2 \cos(kt), \end{aligned}$$

where we have used the fact that all the above series converge in order to manipulate the series as if they were sums. Passing $h \rightarrow 0$, the second term vanishes uniformly, giving

$$\lim_{h \rightarrow 0} \frac{F(a + hr) - F(a)}{h} = \left(t \mapsto 2 \sum_{k \in \mathbb{N}} a_k r_k \cos(kt) \right).$$

By Hölder's inequality, the series $\sum_{k \in \mathbb{N}} |a_k r_k|$ converges, so the fact that the above lies in Y follows as in problem (a).

5(c) We know that if F is differentiable at $a \in X$, then $F'(a)(r) = F'(a; r)$ for any $r \in X$. Thus, it suffices to show that the latter is indeed a derivative.

First, $F'(a; r)$ is clearly linear with respect to r . By Hölder's inequality,

$$|F'(a; r)| \leq 2 \sum_{k \in \mathbb{N}} |a_k r_k| \leq 2 \|a\|_X \|r\|_X,$$

so the map $r \mapsto F'(a; r)$ is indeed a bounded linear operator, with operator norm at most $2\|a\|_X$.

Next, we show that $\|F(a+r) - F(a) - F'(a;r)\|_Y / \|r\|_X \rightarrow 0$ as $r \rightarrow 0$. We have

$$\begin{aligned} |F(a+r) - F(a) - F'(a;r)|(t) &= \left| \sum_{k \in \mathbb{N}} ((a_k + r_k)^2 - a_k^2 - 2a_k r_k) \cos(kt) \right| \\ &\leq \sum_{k \in \mathbb{N}} |r_k^2 \cos(kt)| = \|r\|_X^2. \end{aligned}$$

Since this holds for all $t \in [0, 2\pi]$, it also holds for the supremum over t , yielding

$$\frac{\|F(a+r) - F(a) - F'(a;r)\|_Y}{\|r\|_X} \leq \frac{\|r\|_X^2}{\|r\|_X} = \|r\|_X \rightarrow 0$$

as $r \rightarrow 0$. This proves that F is differentiable at a , and that

$$F'(a)(r)(t) = F'(a;r)(t) = 2 \sum_{k \in \mathbb{N}} a_k r_k \cos(kt) \quad \forall r \in X, t \in [0, 2\pi].$$

Problem 6.

6(a) Consider the vector space $\ell^1(\mathbb{R})$ equipped with the ℓ^1 norm. Let $e_1, e_2, \dots \in \ell^1(\mathbb{R})$ be given by

$$e_1 := (1, 0, 0, 0, \dots), \quad e_2 := (0, 1, 0, 0, \dots), \quad \dots$$

Show that $(e_1, e_2, \dots)$ is a basis for $\ell^1(\mathbb{R})$ (also called a *Schauder basis*). Conclude that $\ell^1(\mathbb{R})$ is separable (that is, it has a subset which is dense and countable).

Hint: For separability, consider the set

$$D := \left\{ \sum_{k=1}^N \beta_k e_k : \beta_1, \dots, \beta_N \in \mathbb{Q}, N \in \mathbb{N} \right\}.$$

6(b) Let U be a *non-separable* normed vector space (that is, U is not separable). Show that U does *not* have a basis.

Solution:

6(a) Let $a \in \ell^1(\mathbb{R})$, $a = (\alpha_1, \alpha_2, \dots)$ where $\|a\|_{\ell^1} = \sum_{k=1}^{\infty} |\alpha_k| < \infty$. Define the partial series

$$a_n := \sum_{k=1}^n \alpha_k e_k = (\alpha_1, \dots, \alpha_n, 0, 0, \dots).$$

Then

$$\|a - a_n\|_{\ell^1} = \sum_{k=n+1}^{\infty} |\alpha_k| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

since the series converges. Hence, $a = \sum_{k=1}^{\infty} \alpha_k e_k$. To see that the

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coefficients are unique, we note that if $u = \sum_{k=1}^{\infty} \beta_k e_k$, then

$$u = (\alpha_1, \alpha_2, \dots) = (\beta_1, \beta_2, \dots),$$

so comparing component by component, we see that $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, etc.

Now let D be as in the hint. It should be clear that D is countable. Let now $a \in \ell^1(\mathbb{R})$; writing

$$a = (\alpha_1, \alpha_2, \dots) = \sum_{k=1}^{\infty} \alpha_k e_k,$$

we let $\beta_{k,n} \in \mathbb{Q}$ (for $n \in \mathbb{N}$, $k \leq n$) be such that $|\beta_{k,n}| \leq |\beta_{k,n+1}|$ for all k, n , and $|\beta_{k,n} - \alpha_k| \leq 1/n^2$. Denote $d_n := \sum_{k=1}^n \beta_{k,n} e_k \in D$. Then

$$\begin{aligned} \|a - d_n\| &= \left\| \sum_{k=1}^{\infty} \alpha_k e_k - \sum_{k=1}^n \beta_{k,n} e_k \right\|_{\ell^1} \\ &= \sum_{k=1}^n |\alpha_k - \beta_{k,n}| + \sum_{k=n+1}^{\infty} |\alpha_k| \\ &\leq \sum_{k=1}^n \frac{1}{n^2} + \sum_{k=n+1}^{\infty} |\alpha_k| \\ &= \frac{1}{n} + \sum_{k=n+1}^{\infty} |\alpha_k|. \end{aligned}$$

The second term vanishes as $n \rightarrow \infty$, so $d_n \rightarrow a$. This proves that D is dense in ℓ^1 , and therefore, that ℓ^1 is separable.

6(b) Assume conversely that U has a basis: There exist $e_1, e_2, \dots \in U$ such that any $u \in U$ can be expressed as

$$u = \sum_{k=1}^{\infty} \alpha_k e_k \tag{*}$$

for unique $\alpha_1, \alpha_2, \dots \in \mathbb{R}$. The same argument as in Problem (a) will prove that U is separable, which is a contradiction.

Define the countable set

$$D := \left\{ \sum_{k=1}^N \beta_k e_k : \beta_1, \dots, \beta_k \in \mathbb{Q}, N \in \mathbb{N} \right\}.$$

We claim that D is dense in U . Indeed, writing some $u \in U$ in the form (*), let $u_n \in D$ for each $n \in \mathbb{N}$ be given by

$$u_n = \sum_{k=1}^n \beta_{k,n} e_k$$

where $\beta_{1,n}, \dots, \beta_{n,n} \in \mathbb{Q}$ are such that $|\beta_{k,n}| \leq |\beta_{k,n+1}|$ for all k, n , and $|\beta_{k,n} - \alpha_n| < 1/(n^2 \|e_k\|)$. Then $u_n \in D$, and

$$\begin{aligned} \|u - u_n\| &= \left\| \sum_{k=1}^{\infty} \alpha_k e_k - \sum_{k=1}^n \beta_{k,n} e_k \right\| \\ &\leq \left\| \sum_{k=n+1}^{\infty} \alpha_k e_k \right\| + \left\| \sum_{k=1}^n (\alpha_k - \beta_{k,n}) e_k \right\| \\ &\leq \left\| \sum_{k=n+1}^{\infty} \alpha_k e_k \right\| + \sum_{k=1}^n \frac{1}{n^2 \|e_k\|} \|e_k\| \\ &= \left\| \sum_{k=n+1}^{\infty} \alpha_k e_k \right\| + \frac{1}{n}. \end{aligned}$$

The first term vanishes as $n \rightarrow \infty$ because the series $(*)$ converges, so we conclude that $u_n \rightarrow u$. But then U has a countable, dense subset, contradicting the assumption of non-separability.

Some information you might find useful**The vector spaces $\ell^p(\mathbb{R})$:**

For $p \in [1, \infty)$, the ℓ^p spaces are defined by

$$\ell^p(\mathbb{R}) := \{a = (\alpha_1, \alpha_2, \dots) : \alpha_1, \alpha_2, \dots \in \mathbb{R}, \|a\|_{\ell^p} < \infty\}$$

where $\|\cdot\|_{\ell^p}$ is the ℓ^p norm,

$$\|a\|_{\ell^p} := \left(\sum_{k \in \mathbb{N}} |\alpha_k|^p \right)^{1/p} \quad \text{for } a = (\alpha_1, \alpha_2, \dots).$$

Hölder's inequality/the Cauchy–Schwarz inequality:

For any $a, b \in \ell^2(\mathbb{R})$, we have

$$\sum_{k \in \mathbb{N}} |\alpha_k \beta_k| \leq \|a\|_{\ell^2} \|b\|_{\ell^2}$$

where $a = (\alpha_1, \alpha_2, \dots)$, $b = (\beta_1, \beta_2, \dots)$.

The space of continuous functions:

$$C([0, 2\pi], \mathbb{R}) := \{f : [0, 2\pi] \rightarrow \mathbb{R} : f \text{ is continuous}\},$$

equipped with the supremum norm,

$$\|f\|_{\sup} := \sup_{t \in [0, 2\pi]} |f(t)|.$$

GOOD LUCK!