UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT2400 — Real Analysis

Day of examination: 8–15 June 2020

This problem set consists of 6 pages.

Appendices: None

Permitted aids: Any

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: There are in total 11 sub-problems (1, 2a, 2b, etc.), and you can get up to 10 points for each sub-problem.

Problem 1

Let X be a normed vector space over $\mathbb{R}$, let $U \subseteq X$ be an open set and let $F\colon U \to \mathbb{R}$ be a (Fréchet) differentiable function.

- We say that a point $a \in U$ is a *critical point* if $F'(a) = \mathbf{0}$ (where $\mathbf{0}$ denotes the zero operator $\mathbf{0}(x) = 0$).
- We say that a point $a \in U$ is a *local minimum* (or *local maximum*) if there is a neighborhood V of a where $F(a) = \min_{x\in V} F(x)$ (or $F(a) = \max_{x\in V} F(x)$).

Show that every local maximum/minimum is a critical point.

Solution: Let $a \in X$ be a local extremum, say, a local maximum. Then $F(a) \geqslant F(x)$ for all x in some neighborhood $B(x;\delta)$. Since F is Fréchet differentiable at a, it is Gateaux differentiable, with derivative

$$F'(a;r) = \lim_{h\to 0} \frac{F(a+hr) - F(a)}{h} \qquad \text{for any } r \in X.$$

The numerator in this expression is non-positive for h small enough, while the denominator h might be of either sign. It follows that the limit must be zero,

$$F'(a;r) = 0 \qquad \forall\, r \in X.$$

Hence, $F'(a)(r) = F'(a;r) = 0$ for all $r \in X$, whence $F'(a) = \mathbf{0}$.

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Problem 2

If X is a metric space and $f\colon X \to \mathbb{R}$ is a function then the *support* of f is the set $\operatorname{supp} f \subseteq X$ defined by

$$\operatorname{supp} f = \overline{A}, \qquad \text{where} \qquad A = \{x \in X \mid f(x) \neq 0\}$$

(and $\overline{A}$ denotes the closure of A). Let $C_c(\mathbb{R}, \mathbb{R})$ be the vector space

$$C_c(\mathbb{R}, \mathbb{R}) = \{f \in C(\mathbb{R}, \mathbb{R}) \mid \operatorname{supp} f \text{ is compact}\},$$

equipped with the supremum norm $\|\cdot\|_{\sup}$. (As usual, $\mathbb{R}$ is equipped with the canonical norm $|\cdot|$.)

2a

Show that every $f \in C_c(\mathbb{R}, \mathbb{R})$ is uniformly continuous.

2b

For any $r \in \mathbb{R}$, define the function $A_r\colon C_c(\mathbb{R}, \mathbb{R}) \to C_c(\mathbb{R}, \mathbb{R})$ by $A_r(f)(t) = f(t + r)$. Show that A_r is a bounded linear operator.

2c

We set now $r = \frac{1}{n}$ for $n \in \mathbb{N}$. Show that the sequence $\{A_{1/n}\}_{n\in\mathbb{N}}$ converges pointwise to I, the identity operator on $C_c(\mathbb{R}, \mathbb{R})$. In other words, show that $\lim\limits_{n\to\infty} A_{1/n}(f) = f$ for all $f \in C_c(\mathbb{R}, \mathbb{R})$.

2d

Show that $A_{1/n}$ does *not* converge to I in operator norm.
Hint: For every $n \in \mathbb{N}$, find a function $f_n \in C_c(\mathbb{R}, \mathbb{R})$ with $\|f\|_{\sup} = 1$ and $\|f_n - A_{1/n}(f_n)\|_{\sup} = 1$.

Solution:

2a

Let $f \in C_c(\mathbb{R}, \mathbb{R})$ and $\varepsilon > 0$. Then $\operatorname{supp} f$ is compact, so there is some bounded interval $[a, b]$ such that $\operatorname{supp} f \subseteq [a, b]$. Since f is continuous on the compact set $[a - 1, b + 1]$, it is uniformly continuous there, so there is some $\delta > 0$ such that $|f(t) - f(s)| < \varepsilon$ when $t, s \in [a - 1, b + 1]$ satisfy $|t - s| < \delta$. If $t, s \in \mathbb{R}$ satisfy $|t - s| < \min(\delta, 1)$ then either $t, s \in [a - 1, b + 1]$, in which case $|f(t) - f(s)| < \varepsilon$, or $t, s \notin [a, b]$, in which case $|f(t) - f(s)| = |0 - 0| < \varepsilon$. Hence, it is uniformly continuous on $\mathbb{R}$.

2b

$A_r(\alpha f + g)(t) = (\alpha f + g)(t + r) = \alpha f(t + r) + g(t + r) = \alpha A_r(f)(t) + A_r(g)(t)$, so A_r is linear. For boundedness, $\|A_r(f)\|_{\sup} = \sup_{t\in\mathbb{R}} |f(t + r)| = \|f\|_{\sup}$, so $\|A\|_{\mathcal{L}} = 1$.

(Continued on page 3.)

2c

Let $f \in C_c(\mathbb{R}, \mathbb{R})$. Then f is uniformly continuous, so for every $\varepsilon > 0$ there is some $\delta > 0$ such that $|f(t+r) - f(t)| \leqslant \varepsilon$ for all $t \in \mathbb{R}$ and $|r| < \delta$. Hence, also $\|A_{1/n}(f) - f\|_{\sup} = \sup_{t\in\mathbb{R}} |f(t+r) - f(t)| \leqslant \varepsilon$ when $|1/n| < \delta$.

2d

We have

$$\|A_{1/n} - I\|_{\mathcal{L}} = \sup_{f \in C_c(\mathbb{R},\mathbb{R}), \|f\|_{\sup}=1} \|A_{1/n}(f) - f\|_{\sup}.$$

We aim to show that $\|A_{1/n} - I\|_{\mathcal{L}} \geqslant 1$ for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, let $f_n \in C_c(\mathbb{R}, \mathbb{R})$ be any function satisfying $f_n(0) = 0$, $f_n(1/n) = 1$, and $\|f\|_{C_c(\mathbb{R},\mathbb{R})} = 1$, such as

$$f_n(t) = \begin{cases} t/n & \text{if } 0 < t < 1/n \\ 1 & \text{if } 1/n \leqslant t \leqslant 1 - 1/n \\ (1-t)/n & \text{if } 1 - 1/n < t < 1 \\ 0 & \text{if } t \geqslant 1 \text{ or } t \leqslant 0. \end{cases}$$

Then $\|A_{1/n}f_n - f_n\|_{\sup} = 1$. Thus,

$$\|A_{1/n} - I\|_{\mathcal{L}} \geqslant \|A_{1/n}f_n - f_n\|_{\sup} = 1.$$

It follows that $A_{1/n} \not\to I$ as $n \to \infty$.

Problem 3

Let $p \in [1, \infty)$ and let $\ell^p(\mathbb{R})$ be the vector space of all sequences $\{a_i\}_{i\in\mathbb{N}}$ in $\mathbb{R}$ such that $\|a\|_{\ell^p} < \infty$, where

$$\|a\|_{\ell^p} = \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p}.$$

3a

For each $a \in \ell^p(\mathbb{R})$, let $F(a) = b$, where b is the sequence with components $b_i = \sin(a_i)$ (for $i \in \mathbb{N}$). Show that $F(a) \in \ell^p(\mathbb{R})$ for every $a \in \ell^p(\mathbb{R})$.

3b

Show that F is Fréchet differentiable, and that the Fréchet derivative is given by

$$F'(a)(r) = \big(\cos(a_i) r_i\big)_{i=1}^{\infty} \qquad \text{for } r \in \ell^p(\mathbb{R}).$$

Solution:

(Continued on page 4.)

3a

Using the estimate $|\sin(a_i)| \leqslant |a_i|$, we get

$$\|F(a)\|_{\ell^p} = \left(\sum_{i=1}^{\infty} |\sin(a_i)|^p\right) \leqslant \left(\sum_{i=1}^{\infty} |a_i|^p\right) = \|a\|_{\ell^p} < \infty.$$

3b

Fix $a \in \ell^p(\mathbb{R})$. We claim that $F'(a) = A$, where $A(r) = \big(\cos(a_i)r_i\big)_{i=1}^{\infty}$. We show first that A is a bounded linear operator from $\ell^p(\mathbb{R})$ into itself. First,

$$\|A(r)\|_{\ell^p(\mathbb{R})} = \left(\sum_{i=1}^{\infty} |\cos(a_i)r_i|^p\right)^{1/p} \leqslant \left(\sum_{i=1}^{\infty} |r_i|^p\right)^{1/p} = \|r\|_{\ell^p(\mathbb{R})} < \infty,$$

so $A(r) \in \ell^p(\mathbb{R})$. It is clear that A is linear. Last, by the above estimate, A is bounded with $\|A\|_{\mathcal{L}} \leqslant 1$. To show that $F'(a) = A$, we observe that a Taylor expansion of $t \mapsto \sin(t)$ yields

$$|\sin(t+h) - \sin(t) - h\cos(t)| \leqslant \frac{h^2}{2} \max_{s\in\mathbb{R}} |\sin''(s)| = \frac{h^2}{2}$$

for any $t, h \in \mathbb{R}$. Next, we note that

$$|r_j| = \big(|r_j|^p\big)^{1/p} \leqslant \left(\sum_{i=1}^{\infty} |r_i|^p\right)^{1/p} = \|r\|_{\ell^p(\mathbb{R})} \qquad \forall\, j \in \mathbb{N}$$

for every $r \in \ell^p(\mathbb{R})$, and hence also $\|r\|_{\ell^\infty(\mathbb{R})} \leqslant \|r\|_{\ell^p(\mathbb{R})}$. Hence,

$$\begin{aligned}\|F(a+r) - F(a) - A(r)\|_{\ell^p} &= \left(\sum_{i=1}^{\infty} \big|\sin(a_i + r_i) - \sin(a_i) - \cos(a_i)r_i\big|^p\right)^{1/p} \\ &\leqslant \left(\sum_{i=1}^{\infty} \big|r_i\big|^{2p}\right)^{1/p} \leqslant \sup_{i\in\mathbb{N}} |r_i| \left(\sum_{i=1}^{\infty} \big|r_i\big|^p\right)^{1/p} \leqslant \|r\|_{\ell^p(\mathbb{R})}^2.\end{aligned}$$

Thus, $F(a+r) - F(a) - A(r) = o(\|r\|_{\ell^p})$ as $r \to 0$, which proves the claim.

Problem 4

Consider the system of equations

$$\begin{cases} x^2 - 3xy + e^{y+z} = -1 \\ -2\cos(x-y)z + zy^4 = 1. \end{cases} \tag{1}$$

Note that $(x, y, z) = (1, 1, -1)$ solves (1).

Show that there is an open interval $(a, b) \subseteq \mathbb{R}$ containing -1 and functions $X, Y\colon (a, b) \to \mathbb{R}$ such that $X(-1) = 1$, $Y(-1) = 1$ and such that $(X(z), Y(z), z)$ solves (1) for every $z \in (a, b)$.

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Solution: Define the function $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ by

$$F(x,y,z) = \begin{pmatrix} x^2 - 3xy + e^{y+z} + 1 \\ -2\cos(x-y)z + zy^4 - 1 \end{pmatrix},$$

and note that $F(1,1,-1) = 0$. The components of F are smooth functions of (x,y,z), so F is continuously differentiable. The Jacobian with respect to (x,y) is

$$J_{(x,y)}F(x,y,z) = \begin{pmatrix} 2x - 3y & -3x + e^{y+z} \\ 2\sin(x-y)z & -2\sin(x-y)z + 4zy^3 \end{pmatrix},$$

so

$$J_{(x,y)}F(1,1,-1) = \begin{pmatrix} -1 & -2 \\ 0 & -4 \end{pmatrix}.$$

This is an invertible matrix, so the bounded linear operator $\frac{\partial F}{\partial(x,y)}(1,1,-1)$ is invertible. The conclusion now follows from the implicit function theorem.

Problem 5

5a

Let $f \in C([-\pi,\pi],\mathbb{C})$ be a 2π-periodic function with Fourier coefficients $\{\alpha_n\}_{n\in\mathbb{Z}}$. Assume that

$$\sum_{n\in\mathbb{Z}} |\alpha_n| < \infty. \tag{2}$$

Show that the Fourier series of f converges uniformly to f.

5b

Conversely, show that if $\{\alpha_n\}_{n\in\mathbb{Z}}$ is some sequence in $\mathbb{C}$ satisfying (2), then the function f defined by

$$f(t) = \sum_{n\in\mathbb{Z}} \alpha_n e^{int} \qquad \text{for } t \in [-\pi,\pi] \tag{3}$$

is well-defined, continuous and 2π-periodic.
Hint: By "well-defined" we mean that the series (3) converges.

5c

Compute the Fourier series of the function $f(t) = t^2$, and explain why the Fourier series converges uniformly to f.
Hint: In 5b, 5c you might need to use what you found in 5a.

Solution:

5a

For each $n \in \mathbb{N}$, let $s_n \in C([-\pi,\pi],\mathbb{C})$ be the trigonometric polynomial $s_n(t) = \sum_{|k|\leqslant n} \alpha_k e^{ikt}$. Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that

(Continued on page 6.)

$\sum_{|k|\geqslant N} |\alpha_k| < \varepsilon$. Then for every $k, l \geqslant N$,

$$|s_k(t) - s_l(t)| \leqslant \sum_{|m|\geqslant N} |\alpha_m| < \varepsilon.$$

Since the choice of N is independent of t, this proves that $\{s_n\}_{n\in\mathbb{N}}$ converges uniformly to some function g, and since the convergence is uniform, g is continuous on $[-\pi, pi]$. The limit g is 2π-periodic since each s_k is: $g(-\pi) = \lim_{n\to\infty} s_n(-\pi) = \lim_{n\to\infty} s_n(\pi) = g(\pi)$. On the other hand, since f is continuous and 2π-periodic, we know that $\{s_n\}_{n\in\mathbb{N}}$ converges in Césaro mean to f. This proves that $f = g$.

5b

By 5a, the series (3) converges uniformly, so f is well-defined. Since each partial sum is a trigonometric polynomial, and hence is both continuous and 2π-periodic, and the convergence is uniform, the same is true for the limit.

5c

We compute $\alpha_n = \int_{-\pi}^{\pi} t^2 e^{-int}\,dt$. If $n = 0$ then

$$\alpha_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 = \frac{\pi^2}{3}.$$

For $n \neq 0$ we get from repeated integration by parts

$$\begin{aligned}
\alpha_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 e^{-int}\,dt = \underbrace{\left[-\frac{1}{2\pi i n} t^2 e^{-int}\right]_{t=-\pi}^{\pi}}_{=0} + \frac{2}{2\pi i n}\int_{-\pi}^{\pi} t e^{-int}\,dt \\
&= \left[-\frac{1}{\pi(in)^2} t e^{-int}\right]_{t=-\pi}^{\pi} + \underbrace{\frac{1}{\pi(in)^2}\int_{-\pi}^{\pi} e^{-int}\,dt}_{=0} \\
&= \frac{2(-1)^n}{n^2}.
\end{aligned}$$

Hence, the Fourier series of f is

$$\frac{\pi^2}{3} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{2(-1)^n}{n^2} e^{int}.$$

The series (2) converges: Since $|\alpha_n| \leqslant \frac{2}{n^2}$ for $n \neq 0$ we get

$$\sum_{n\in\mathbb{Z}} |\alpha_n| \leqslant \frac{\pi^2}{3} + 2\sum_{n\in\mathbb{N}} \frac{2}{n^2} < \infty$$

since $\sum_{n\in\mathbb{N}} \frac{1}{n^2}$ converges. By problem 5a, we conclude that the Fourier series converges uniformly.