UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT2400 — Real Analysis

Day of examination: 18 August 2021

Examination hours: 09:00 – 13:00

This problem set consists of 5 pages.

Appendices: None

Permitted aids: Any

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: There are in total 10 sub-problems, and you can get up to 10 points for each sub-problem, for a total of 100 points.

Problem 1. (10 points)
Consider $\mathbb{R}^2$ equipped with the Euclidean norm $\|x\| = \sqrt{x_1^2 + x_2^2}$ and the corresponding metric $d(x, y) = \|x - y\|$. Define the unit circle $S = \{x \in \mathbb{R}^2 : \|x\| = 1\}$. Explain why

- (S, d) is a metric space
- $(S, \|\cdot\|)$ is *not* a normed vector space.

Note: You may use the fact that $(\mathbb{R}^2, \|\cdot\|)$ is a normed vector space.

Solution: S is a closed subset of $\mathbb{R}^2$, and a closed subset of a metric space always gives rise to a new metric space. Thus, (S, d) is a metric space.

$(S, \|\cdot\|)$ is not a normed vector space because S is not a vector space: If $x \in S$ then $2x \notin S$.

Problem 2. (10 points)
Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{-x^2}$ is uniformly continuous.

Solution: We first note that f is continuous on $\mathbb{R}$, since it is the composition of the continuous functions $x \mapsto e^x$ and $x \mapsto -x^2$.

Let $\varepsilon > 0$ and let $M > 0$ be such that $|f(M - \varepsilon)| < \varepsilon$. (Such a number exists because $f(x) \to 0$ as $x \to \infty$.) Next, let $\delta > 0$ be such that $|x - y| < \delta$ and $x, y \in [-M, M]$ implies $|f(x) - f(y)| < \varepsilon$ (such a δ exists because f is continuous, and hence uniformly continuous on the compact set $[-M, M]$). If now $x, y \in \mathbb{R}$ with $|x - y| < \delta$ then either

- $x, y \in [-M, M]$,
- $x, y \notin [-M + \delta, M - \delta]$.

(Continued on page 2.)

In the first case we already know that $|f(x) - f(y)| < \varepsilon$. In the second case we also know that $|x|, |y| \geqslant M - \varepsilon$, so $|f(x)| \leqslant |f(M-\varepsilon)| < \varepsilon$, and likewise for y. Since $f(x), f(y) > 0$ we then get $|f(x) - f(y)| < \varepsilon$.

Alternatively: We note that f is Lipschitz: $\frac{df}{dx}(x) = -2xe^{-x^2}$, which is a bounded function (its extrema lie at $x = \pm 1/\sqrt{2}$, where $|\frac{df}{dx}(x)| = \sqrt{2}e^{-1/2}$), so in particular, f is Lipschitz. This concludes the proof, since every Lipschitz function is also uniformly continuous.

Problem 3. (20 points)
Fix some $p \in [1, \infty]$ and let $x \in \ell^p(\mathbb{R})$ with components $x = (x(1), x(2), \dots)$. Define $y_n = \big(x(1), x(2), \dots, x(n), 0, 0, 0, \dots\big)$ for every $n \in \mathbb{N}$, so that

$$y_1 = \big(x(1), 0, 0, 0, \dots\big), \quad y_2 = \big(x(1), x(2), 0, 0, 0, \dots\big),$$

and so on.

(a) Explain why $y_n \in \ell^p(\mathbb{R})$ for every $n \in \mathbb{N}$. Show that if $p < \infty$ then $\{y_n\}_{n\in\mathbb{N}}$ is Cauchy.

(b) Is the same true for $p = \infty$? If yes, prove it. If no, provide a counterexample.

Solution:

(a) Since y_n has only finitely many nonzero elements, its ℓ^p norm is automatically finite. If $x \in \ell^p$ then $\sum_{i=1}^{\infty} |x(i)|^p < \infty$. For an $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that $\sum_{i=N}^{\infty} |x(i)|^p < \varepsilon$. If $m, n \geqslant N$ and, say, $m < n$ then

$$\|y_n - y_m\|_{\ell^p}^p = \sum_{i=m+1}^{n} |x(i)|^p \leqslant \sum_{i=N}^{\infty} |x(i)|^p < \varepsilon.$$

Hence, $\{y_n\}_n$ is Cauchy.

(b) No. Let $x = (1, 1, 1, \dots)$. Then $\|x\|_{\ell^\infty} = 1$, so $x \in \ell^\infty$, but if $n \neq m$ then

$$\|y_n - y_m\|_{\ell^\infty} = \|(0, \dots, 0, 1, \dots, 1, 0, \dots)\|_{\ell^\infty} = 1.$$

Hence, $\{y_n\}_n$ cannot possibly be Cauchy.

Problem 4. (10 points)
Let $X = C([0,1], \mathbb{R})$ be equipped with the supremum metric $d_\infty(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)|$. Let $L\colon X \to \mathbb{R}$ be a continuous function satisfying

$$L(p) = 0 \qquad \text{for all polynomials } p.$$

Show that $L(f) = 0$ for all $f \in X$.

Solution: Let $f \in X$. By Weierstrass' approximation theorem, there is a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ such that $p_n \to f$ as $n \to \infty$. Since L is continuous, we get $L(f) = \lim_{n\to\infty} L(p_n) = \lim_{n\to\infty} 0 = 0$.

Problem 5. (20 points)
Let $X = C([0,2], \mathbb{R})$, equipped with the supremum norm $\|f\|_\infty =$

(Continued on page 3.)

$\sup_{t\in[0,2]} |f(t)|$, and let $Y = C([0,1],\mathbb{R})$, equipped with the L^1 norm $\|g\|_1 = \int_0^1 |g(t)|\,dt$.

(a) Show that $L\colon X \to Y$, defined by

$$L(f) = g, \qquad \text{where } g(t) = f(2t)\ \forall t \in [0,1],$$

is a bounded linear functional.

(b) Show that L is bijective, but that the inverse $L^{-1}\colon Y \to X$ is unbounded.

Solution:

(a) For linearity, we have for any $\alpha \in \mathbb{R}$ and $f, g \in X$

$$\begin{aligned} L(\alpha f + g)(t) &= (\alpha f + g)(2t) = \alpha f(2t) + g(2t) \\ &= \alpha L(f)(t) + L(g)(t) = (\alpha L(f) + L(g))(t) \end{aligned}$$

for all t, so $L(\alpha f + g) = \alpha L(f) + L(g)$.

For boundedness, we have

$$\|L(f)\|_1 = \int_0^1 |f(2t)|\,dt \leqslant \int_0^1 \|f\|_\infty\,dt = \|f\|_\infty,$$

so L is bounded with $\|L\|_{\mathcal{L}} \leqslant 1$.

(b) L is clearly bijective, since $L(f) = g$ if and only if $f(t) = g(t/2)$ $\forall t \in [0,2]$. To see that L^{-1} is unbounded, let $\{g_n\}_{n\in\mathbb{N}}$ be the sequence

$$g_n(t) = \begin{cases} n - n^2 t & 0 \leqslant t < 1/n \\ 0 & 1/n \leqslant t \leqslant 1. \end{cases}$$

Then $\|g_n\|_1 = 1$ for all $n \in \mathbb{N}$, while $\|f_n\|_\infty = g_n(0) = n$. Hence, there exists no constant $C > 0$ such that $\|L^{-1}(g_n)\|_\infty \leqslant C\|g_n\|_1$, so L^{-1} is unbounded.

Problem 6. (20 points)
Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$F(x) = \begin{pmatrix} x_1^3(x_2+1) - x_2^2 \\ x_1 - x_2^5 \end{pmatrix} \qquad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2.$$

(a) Show that F is everywhere Fréchet differentiable, and that

$$F'(x)(h) = \begin{pmatrix} 3x_1^2(x_2+1) & x_1^3 - 2x_2 \\ 1 & -5x_2^4 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \qquad \forall x, h \in \mathbb{R}^2.$$

(b)

(i) It's difficult, or impossible, to find a formula for F^{-1} (if it exists at all). Why?

(Continued on page 4.)

(ii) Let

$$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and note that $F(b) = c$. Show that there is a neighbourhood U of b and a neighbourhood V of c such that $F\colon U \to V$ is bijective. Compute $(F^{-1})'(c)$.

Solution:

(a) A function $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ is Fréchet differentiable if its Jacobian exists and is continuous everywhere, in which case $F'(x)(h) = DF(x)h$. Its Jacobian is

$$DF(x) = \begin{pmatrix} 3x_1^2(x_2+1) & x_1^3 - 2x_2 \\ 1 & -5x_2^4 \end{pmatrix}.$$

Each entry is a polynomial, so DF is continuous, and $F'(x)(h) = DF(x)h$.

(b) Inverting F would involve inverting a system of fifth order polynomials, which in general is impossible.

We have

$$F'(b)(h) = \begin{pmatrix} 6 & -1 \\ 1 & -5 \end{pmatrix} h$$

which is invertible, since the Jacobian is invertible:

$$DF(b)^{-1} = \frac{1}{-29} \begin{pmatrix} -5 & 1 \\ -1 & 6 \end{pmatrix}.$$

The inverse function theorem then guarantees the existence of neighbourhoods U, V such that $F\colon U \to V$ is bijective and

$$(F^{-1})'(c)(h) = (F'(b))^{-1}(h) = \frac{1}{-29} \begin{pmatrix} -5 & 1 \\ -1 & 6 \end{pmatrix} h.$$

Problem 7. (10 points)
A bounded linear operator $L\colon X \to Y$ is *compact* if $\{L(u_n)\}_{n\in\mathbb{N}}$ has a convergent subsequence whenever $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in X.

Let $X = Y = C([-\pi,\pi],\mathbb{R})$ equipped with the supremum metric, and for some fixed $N \in \mathbb{N}$, let L be the Fourier projection

$$L(u)(t) = \sum_{n=-N}^{N} \alpha_n e^{int} \qquad \forall t \in [-\pi,\pi],\ u \in X$$

where $\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(s)e^{-ins}\,ds$ is the nth Fourier coefficient of u. Show that L is a bounded, compact operator. (You do not need to show that L is linear.)

Hint: Apply the Arzela–Ascoli theorem.

(Continued on page 5.)

Solution: L is bounded: We have $|\alpha_n| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(s)||e_n(-s)|\,ds \leqslant \|u\|_\infty$, so

$$\|L(u)\|_\infty \leqslant \sum_{n=-N}^{N} |\alpha_n|\|e_n\|_\infty \leqslant (2N+1)\|u\|_\infty.$$

Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in X. We claim that $\{L(u_n)\}_{n\in\mathbb{N}}$ is equicontinuous. Indeed,

$$\left|\frac{d}{dt}L(u_n)(t)\right| = \left|\sum_{n=-N}^{N} \alpha_n ine^{int}\right| \leqslant \sum_{n=-N}^{N} \|u\|_\infty n\|e_n\|_\infty \leqslant (2N+1)N\|u\|_\infty,$$

so $\{L(u_n)\}_n$ is uniformly Lipschitz, and hence equicontinuous. It follows that $\{L(u_n)\}_n$ is a bounded, equicontinuous sequence in $Y = C([0,1],\mathbb{R})$, so by Arzela–Ascoli, it has a convergent subsequence. We conclude that L is compact.