As you will have noticed, most of the results in the textbook are not proved. This is a conscious decision by the author — he wants to teach the students to make their own proofs. This plan depends on the fact that many of the proofs in volume 2 are very similar to proofs in volume 1. As we have not covered volume 1, the plan is overly ambitious in our case, and I have therefore decided to provide proofs of many (but far from all) the results in the book. I have concentrated on the results that are essential for the understanding of the theory, leaving many of the more “applied” results for the problem sessions.

**Notation:** I shall use the traditional notation \( \{x_n\} \) for sequences instead of the author’s choice \( (x^{(n)}) \).

**Proposition 12.1.20**

If the sequence \( \{x_n\} \) converges to both \( x \) and \( x' \), then \( \lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x_n, x') = 0 \). By the Triangle Inequality and Symmetry

\[
d(x, x') \leq d(x, x_n) + d(x_n, x') \to 0 + 0 = 0
\]

and hence \( d(x, x') = 0 \). By Positivity, \( x = x' \).

**Proposition 12.2.10**

It suffices to prove \( (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a) \) as we can then prove all implications by moving around the circle.

\( (a) \Rightarrow (b) \): Assume that \( x_0 \) is an adherent point of \( E \). Then \( B(x_0, r) \cap E \neq \emptyset \) for all \( r \). Consequently, \( x_0 \) is not an exterior point, and the only remaining options are that \( x_0 \) is an interior point or a boundary point.

\( (b) \Rightarrow (c) \): Assume that \( x_0 \) is an interior point or a boundary point. In either case, the set \( B(x_0, \frac{1}{n}) \cap E \) is nonempty and contains an element \( x_n \). Since \( d(x_0, x_n) < \frac{1}{n} \), we have a sequence \( \{x_n\} \) of points from \( E \) converging to \( x_0 \).

\( (c) \Rightarrow (a) \): Assume that \( \{x_n\} \) is a sequence from \( E \) converging to \( x_0 \). Then for every \( r > 0 \), there is an \( N \in \mathbb{N} \) such that \( d(x_0, x_n) < r \) for all \( n \geq N \). This implies that \( B(x_0, r) \cap E \) is nonempty for all \( r > 0 \), and hence \( x_0 \) is an adherent point of \( E \).
Lemma 12.4.3
Given an $\epsilon > 0$, we must show that there is an $J \in \mathbb{N}$ such that $d(x_{n_j}, x_0) < \epsilon$ whenever $j \geq J$. Since the original sequence $\{x_n\}$ converges to $x_0$, there is an $N \in \mathbb{N}$ such that $d(x_n, x_0) < \epsilon$ whenever $n \geq N$. Choose $J$ such that $n_j \geq N$. Then $d(x_{n_j}, x_0) < \epsilon$ whenever $j \geq J$ (since $n_j \geq n, j \geq N$ whenever $j \geq J$).

Proposition 12.4.5
Assume first that $L$ is a limit point of $\{x_n\}$. Define an increasing sequence $n_1 < n_2 < \ldots < n_k < \ldots$ of natural numbers as follows: $n_1$ is the smallest number in $\mathbb{N}$ such that $d(x_{n_1}, L) < 1$; $n_2$ is the smallest number larger than $n_1$ such that $d(x_{n_2}, L) < 1/2$; $n_3$ is the smallest number larger than $n_2$ such that $d(x_{n_3}, L) < 1/3$ etc. Since $L$ is a limit point, it is always possible to continue this procedure, and the subsequence $\{x_{n_j}\}$ converges to $L$ since $d(x_{n_j}, L) < 1/j$.

Assume next that $\{x_{n_j}\}$ is a subsequence converging to $L$. To show that $L$ is a limit point, we must show that for any $\epsilon > 0$ and $N \in \mathbb{N}$, there in an $n \geq N$ such that $d(x_n, L) < \epsilon$. Since the subsequence $\{x_{n_j}\}$ converges to $L$, we can find a $J \in \mathbb{N}$ such that $d(x_{n_j}, L) < \epsilon$ whenever $j \geq J$. Choose $n = n_j$ where $j \geq J$ and $n_j \geq N$. Then $n \geq N$, $d(x_n, L) < \epsilon$ and hence $L$ is a limit point.

Lemma 12.4.7
We must show that for each $\epsilon < 0$ there is an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$. Since $\{x_n\}$ converges to $x_0$, there is an $N \in \mathbb{N}$ such that $d(x_n, x_0) < \frac{\epsilon}{2}$ for all $n \geq N$. If $n, m \geq N$, we thus have

$$d(x_n, x_m) \leq d(x_n, x_0) + d(x_0, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

by the Triangle Inequality, and hence $\{x_n\}$ is a Cauchy sequence.

Lemma 12.4.9
We must show that for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $d(x_n, x_0) < \epsilon$ whenever $n \geq N$. Since $\{x_n\}$ is a Cauchy sequence, there is an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon/2$ for all $n, m \geq N$. Since the subsequence converges to $x_0$, there is an $n_j \geq N$ such that $d(x_{n_j}, x_0) < \epsilon/2$. By the Triangle Inequality, we have

$$d(x_n, x_0) \leq d(x_n, x_{n_j}) + d(x_{n_j}, x_0) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n \geq N$. 
Proposition 12.5.5

We shall argue contrapositively. First we assume that \( X \) is unbounded, and prove that it is not compact; then we assume that \( X \) is not complete, and prove that it is not compact.

Assume that \( X \) is unbounded. Pick a point \( x \in X \), then all the sets \( B(x, n) \) are nonempty, and if we pick a point \( x_n \) in each, we get a sequence such that \( d(x, x_n) \to \infty \). For any subsequence \( \{x_{n_j}\} \), we must also have \( d(x, x_{n_j}) \to \infty \). It suffices to prove that no such subsequence can converge. Assume for contradiction that it does converge to some point \( y \). Then by the Triangle Inequality we have

\[
d(x, x_{n_j}) \leq d(x, y) + d(y, x_{n_j}) \to d(x, y)
\]

which is absurd since \( d(x, x_{n_j}) \to \infty \).

Assume now that \( X \) is not complete, and let \( \{x_n\} \) be a Cauchy sequence that does not converge. If \( \{x_n\} \) had a subsequence converging to some point \( x \), then \( \{x_n\} \) would also converge to \( x \) according to Lemma 12.4.9. Hence \( \{x_n\} \) cannot have a convergent subsequence, and consequently \( X \) is not compact.

Corollary 12.5.6

Boundedness follows immediately from Proposition 12.5.5, and closedness follows from the same proposition pluss the following argument which shows that only closed subspaces are complete: If \( Y \) is not closed, choose a point \( a \in Y \setminus X \). Then there is a sequence \( \{y_n\} \) from \( Y \) converging to \( a \). This is a Cauchy sequence in \( Y \) that does not converge to a point in \( Y \), and hence \( Y \) is not complete.

Theorem 12.5.7

We already know that a compact set has to be closed and bounded, and only need to prove that a closed and bounded subset \( E \) of \( \mathbb{R}^m \) is always compact. Let \( d \) be any of the three metrics in the theorem. We must show that a sequence \( \{x_n\} \) of points in \( E \) always has a subsequence converging in the \( d \)-metric to a point in \( E \).

Since \( E \) is bounded, we can find numbers \( a, b \) such that \( E \) is contained in the \( m \) dimensional cube \( K_0 = [a, b]^m \). We can cut \( K_0 \) into \( 2^m \) closed cubes whose sides are half the length of the sides of \( K_0 \). At least one of these smaller cubes must contain infinitely many of the terms in the sequence \( \{x_n\} \). Call this cube \( K_1 \) (if there is more than one such cube, just choose one of them). We now cut \( K_1 \) into \( 2^m \) smaller cubes in exactly the same way, and find a new cube \( K_2 \) whose sides are half the length of the sides of \( K_1 \), and which contains infinitely many terms of the sequence.
Continuing the procedure, we get a nested sequence of \( m \)-dimensional cubes \( \{ K_n \} \), each half the size of the previous, and each containing infinitely many of the elements in the sequence \( \{ x_n \} \). To construct a convergent subsequence, we choose natural numbers \( n_1 < n_2 < \ldots < n_j < \ldots \) by the following procedure: Let \( n_1 \) be the smallest number such that \( x_{n_1} \in K_1 \); let \( n_2 \) be the smallest number larger than \( n_1 \) such that \( x_{n_2} \in K_2 \); let \( n_3 \) be the smallest number larger than \( n_2 \) such that \( x_{n_3} \in K_3 \) etc. Since each cube \( K_n \) contains infinitely many terms from the sequence, it is always possible to carry out this procedure.

We now have a subsequence \( \{ x_{n_j} \} \) such that \( x_{n_j} \in K_j \) for each \( j \). As the size of the cubes decreases to zero, \( \{ x_{n_j} \} \) is a Cauchy sequence in the \( d \)-metric (regardless of which of the metrics in the theorem \( d \) is), and since \( \mathbb{R}^m \) is complete, \( \{ x_{n_j} \} \) converges to a point \( x \in \mathbb{R}^m \). Since \( E \) is closed, \( x \in E \).

This shows that any sequence from \( E \) has a subsequence which converges to a point in \( E \), and hence \( E \) is compact.

**Corollary 12.5.9**

Assume for contradiction that \( \bigcap_{n=1}^{\infty} K_n = \emptyset \). Then the family \( \{ K_n^c \}_{n \in \mathbb{N}} \) forms an open cover of the compact set \( K_1 \) (in fact, they cover all of \( X \)), and according to Theorem 12.5.8, there must be a finite subcover \( K_n^c_1, \ldots, K_n^c_k \) where \( n_1 < n_2 < \ldots < n_k \). Since the original sequence \( \{ K_n \} \) is decreasing, the sequence \( \{ K_n^c \} \) of complements is increasing, and hence \( K_n^c_{n_k} = K_n^c_1 \cup K_n^c_2 \cup \ldots \cup K_n^c_{n_k} \supset K_1 \). But this is impossible since \( K_{n_k} \) is a nonempty subset of \( K_1 \).

**Theorem 12.5.10a**

We already know that a compact set needs to be closed, and hence it suffices to prove that a closed subset \( Z \) of a compact set \( Y \) is compact. Assume that \( \{ x_n \} \) is a sequence in \( Z \). Since \( Z \subset Y \) and \( Y \) is compact, \( \{ x_n \} \) has a subsequence converging to a point \( y \in Y \). Since \( Z \) is closed, \( y \in Z \), and hence any sequence in \( Z \) has a subsequence converging to an element in \( Z \). This means that \( Z \) is compact.