MAT2400: Mandatory Assignment I, Spring 2014: Solution

Problem 1. We use induction on n. For $n = 1$ the statement is obvious, and for $n = 2$ it is the triangle inequality. Assume that the statement holds for $n = k$, i.e.

$$d(x_1, x_k) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{k-1}, x_k)$$

Adding $d(x_k, x_{k+1})$ on both sides, we get

$$d(x_1, x_k) + d(x_k, x_{k+1}) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{k-1}, x_k) + d(x_k, x_{k+1})$$

By the triangle inequality

$$d(x_1, x_{k+1}) \leq d(x_1, x_k) + d(x_k, x_{k+1})$$

and hence

$$d(x_1, x_{k+1}) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{k-1}, x_k) + d(x_k, x_{k+1})$$

This shows that the statement holds for $n = k + 1$ and concludes the induction argument.

Problem 2. a) For $0 \leq x < 1$, $x^n \to 0$ as $n \to \infty$, and hence

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1-x}{1+x^n} = \frac{1-x}{1+0} = 1 - x$$

for such x. For $x > 1$, $x^n \to \infty$ as $n \to \infty$, and hence

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1-x}{1+x^n} = 0$$

for such x. Finally, for $x = 1$, $f_n(x) = 0$ for all n and hence $\lim_{n\to\infty} f_n(x) = 0 = 1 - x$ also in this case.

b) Each f_n is differentiable at $x = 1$ as we can use the quotient rule to compute the derivative. The limit function is not differentiable at $x = 1$ as

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x-1} = \lim_{x\to 1^-} \frac{1-x-0}{x-1} = -1$$

while

$$\lim_{x\to 1^+} \frac{f(x) - f(1)}{x-1} = \lim_{x\to 1^-} \frac{0-0}{x-1} = 0$$

and hence the *two-sided* limit $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$ does not exist.

c) For $x \in [0, 1]$, we have

$$|f(x) - f_n(x)| = |(1-x) - \frac{1-x}{1+x^n}| = (1-x)|1 - \frac{1}{1+x^n}| =$$

$$= (1-x)\left|\frac{1+x^n}{1+x^n} - \frac{1}{1+x^n}\right| = (1-x)\frac{x^n}{1+x^n} \leq (1-x)x^n$$

where we in the last step have used that $1 + x^n \geq 1$.

Here are two different proofs of uniform continuity:

Alternative I: Assume that $\epsilon > 0$ is given and choose $N \in \mathbb{N}$ so large that $(1-\epsilon)^N < \epsilon$. For $n \geq N$, we then have

$$|f(x) - f_n(x)| \leq (1-x)x^n < \begin{cases} (1-x)(1-\epsilon)^n < \epsilon & \text{when } 0 \leq x < 1-\epsilon \\ \\ \epsilon x^n \leq \epsilon & \text{when } 1-\epsilon \leq x \leq 1 \end{cases}$$

This shows that $\{f_n\}$ converges uniformly to f on $[0,1]$.

Alternative II: Another way to prove uniform convergence, is to show that

$$\sup\{|f(x) - f_n(x)| \ : \ x \in [0,1]\} \to 0 \quad \text{as } n \to \infty$$

Since $|f(x) - f_n(x)| \leq (1-x)x^n$, it suffices to show that

$$\sup\{(1-x)x^n \ : \ x \in [0,1]\} \to 0 \quad \text{as } n \to \infty$$

The last supremum we can find by differentiating $g_n(x) = (1-x)x^n = x^n - x^{n+1}$. We get $g_n'(x) = nx^{n-1} - (n+1)x^n$, and it's easy to use this to check that the maximum of g_n is at $x = \frac{n}{n+1}$, and that the maximum value is $g_n(\frac{n}{n+1}) = (1 - \frac{n}{n+1})(\frac{n}{n+1})^n = \frac{1}{n+1}(\frac{n}{n+1})^n < \frac{1}{n+1}$. Hence

$$\sup\{|f(x) - f_n(x)| \ : \ x \in [0,1]\} \leq \sup\{(1-x)x^n \ : \ x \in [0,1]\} \leq \frac{1}{n+1} \to 0$$

as $n \to \infty$. This proves uniform convergence.

Problem 3. a) The idea is to keep the track and change the parametrization. Note that the function

$$u(t) = a + \frac{b-a}{d-c}(t-c)$$

maps the interval $[c, d]$ continuously onto $[a, b]$ (the graph of u is just the straight line through the points (c, a) and (d, b).) Hence

$$\hat{r}(t) = r(u(t)), \quad t \in [c, d],$$

is a continuous path that runs through exactly the same values as

$$r(t), \quad t \in [a, b],$$

and the two paths connect the same points. (Note that $\hat{r}$ is continuous as it is the composition of the two continuous functions r and u.)

b) We have to check the three properties of an equivalence relation:

(i) *Reflexivity:* We have $x \sim x$ since the path that stands still at x connects x to x (i.e. the path $r(t) = x$ for all $x \in [a, b]$).

(ii) *Symmetry:* Assume that $r : [a, b] \to X$ is a path connecting x to y. Then $\tilde{r}(s) = r(b + a - s))$, $s \in [a, b]$, is a path connecting y to x ($\tilde{r}$ is just r run backwards). Hence $x \sim y$ implies $y \sim x$.

(iii) *Transitivity:* Assume $x \sim y$ and $y \sim z$. Then there are paths r_1 and r_2 connecting x to y and y to z, respectively. By a) we may choose the parametrizations such that $r_1 : [0, 1] \to X$ and $r_2 : [1, 2] \to X$, and hence $r : [0, 2] \to X$ defined by

$$r(t) = \begin{cases} r_1(t) & \text{if } t \in [0, 1] \\ \\ r_2(t) & \text{if } t \in [1, 2] \end{cases}$$

is a path connecting x to z, i.e. $x \sim z$ (note that r is continuous at 1 since $r_1(1) = r_2(1)$ and r_1 and r_2 are continuous).

c) Assume that P is a component and $x, y \in P$. We must show that there is a path from x to y *that lies entirely in* P. Since x and y belong to the same component P, $x \sim y$ and hence there is a path $r : [a, b] \to X$ *in* X that connects x to y. Any point z on this path is connected to x by a part of the path, and hence $z \in P$. (More precisely: Since z is on the path, $z = r(c)$ for some $c \in [a, b]$, and hence $r : [a, c] \to X$ is a path connecting x to z). This shows that the path r lies entirely in P, and hence P is path-connected.

d) Pick $x \in C$ and let $[x]$ be the component of x. If y is in C, there is a path in C connecting x to y (here we are using that C is path-connected), and consequently $y \in [x]$. Hence C is a subset of the component $[x]$.

e) Let $u, v \in f(C)$, then there exist points $x, y \in C$ such that $f(x) = u$ and $f(y) = v$. Since C is path-connected, there is a path r in C connecting x and y, and $\hat{r}(t) = f(r(t))$ is a path in $f(C)$ connecting u and v (Note that $\hat{r}$ is continuous as it is the composition of the two continuous functions f and r.)

f) Assume for contradiction that $\{x, y\}$ is path-connected. There must then be a path r in $\{x, y\}$ connecting x to y. Let $s = \inf\{t : r(t) = y\}$. Since r is a path in $\{x, y\}$, $r(s)$ is either x or y.

Assume first that $r(s) = y$. Then $s \neq 0$ (since $r(0) = x$) and $x(t) = x$ for all $t < s$. If $\{t_n\}$ is a sequence increasing to t, we get $y = r(s) = \lim_{n\to\infty} r(t_n) = \lim_{n\to\infty} x = x$ by continuity of r. This is a contradiction.

Assume next that $r(s) = x$. Then $s \neq 1$ (since $r(1) = y$), and by definition of s there must for each $n \in \mathbb{N}$ be a point t_n, $s \leq t_n < s + \frac{1}{n}$, such that $r(t_n) = y$. But then $x = r(s) = \lim_{n\to\infty} r(t_n) = \lim_{n\to\infty} y = y$ by continuity of r. This is again a contradiction, and since there are no other alternatives, $\{x, y\}$ cannot be path-connected.

g) Here is the silliest example: Let $X = \{x, y\}$, $Y = \{u\}$ both have the discrete metric, and let $f : X \to Y$ be the only function from X to Y. Then $X = f^{-1}(Y)$ is *not* path-connected according to f) above, but Y obviously is.

To get a slightly less silly version of the same idea, let $X = Y = \mathbb{R}$ and put $f(x) = x^2$. Then $D = \{1\}$ is path-connected, but $f^{-1}(D) = \{-1, 1\}$ is not according to f).