MAT 2400: Mandatory assignment 2, S-15'

Deadline*: You must turn in your paper before 2.30 p.m., Thursday, April 30., 2015, in the designated area on the 7th floor of NHA. Remember to use the official front page available on the 7th floor and at*

http://www.mn.uio.no/math/studier/admin/obligatorisk-innlevering/obligforside.pdf

If you due to illness or other circumstances want to extend the deadline, you must apply for an extension to studieinfo@math.uio.no Remember that illness has to be documented by a medical doctor! See

http://www.mn.uio.no/math/studier/admin/obligatorisk-innlevering/index.html

for more information about the rules for mandatory assignments.

Instructions*: The assignment is compulsory, and students who do not get their paper accepted, will not get access to the final exam. To get the assignment accepted, you need a score of at least 60%. In the evaluation, credit will be given for a clear and well-organized presentation. All questions (points 1a), 1b) etc.) have equal weight. Students who do not get their original paper accepted, but who have made serious and documented attempts to solve the problems, will get one chance of turning in an improved version.*
In solving the problems you may collaborate with others and use tools of all kinds. However, the paper you turn in should be written by you *(by hand or computer) and should reflect your understanding of the material. If we are not convinced that you understand your own paper, we may ask you to give an oral presentation.*

Problem 1. Let (X, d) be a bounded metric space, and let $P(X)$ denote the collection of non-empty closed subsets of X. For A and B in $P(X)$, let

$$h(A, B) = \sup_{x \in X} |\text{dist}(x, A) - \text{dist}(x, B)|,$$

where $\text{dist}(x, C)$ is given by

$$\text{dist}(x, C) = \inf_{c \in C} d(x, c).$$

The function h is called *the Hausdorff metric.*

a) Show that if $h(A, B) = 0$ then $A = B$. Here A and B are two non-empty closed subsets of X.

Possible answer: Assume that $h(A, B) = 0$, we must show that $A = B$. For any $a \in A$,

$$\begin{aligned} 0 &= \sup_{x \in X} |\text{dist}(x, A) - \text{dist}(x, B)| \\ &\geq |\text{dist}(a, A) - \text{dist}(a, B)| \\ &= \text{dist}(a, B), \end{aligned}$$

which means that $a \in B$, since B is closed. In the same way, for any $b \in B$, we get that $b \in A$. Therefore $A = B$.

b) Show that h is a metric on $P(X)$.

Possible answer: We obviously have that $h(A, A) = 0$, and part **b**) says that $h(A, B) = 0 \Rightarrow A = B$. The symmetry $h(A, B) = h(B, A)$ is immediate. It remains to show the triangle inequality.

$$\begin{aligned} h(A, C) &= \sup_{x\in X} |\mathrm{dist}(x, A) - \mathrm{dist}(x, C)| \\ &\le \sup_{x\in X} |\mathrm{dist}(x, A) - \mathrm{dist}(x, B)| + \sup_{x\in X} |\mathrm{dist}(x, B) - \mathrm{dist}(x, C)| \\ &= h(A, B) + h(B, C). \end{aligned}$$

c) For A and B in $P(X)$, let $\hat{h}$ be defined as

$$\hat{h}(A, B) = \max\left\{\sup_{a\in A} \mathrm{dist}(a, B), \sup_{b\in B} \mathrm{dist}(b, A)\right\}.$$

Show that

$$\hat{h}(A, B) = h(A, B) \quad \text{for all } A,\ B \text{ in } P(X).$$

(**Hint**: Show the two inequalities $h(A, B) \ge \hat{h}(A, B)$ and $\hat{h}(A, B) \ge h(A, B)$.)

Possible answer: Let $b \in B$,

$$h(A, B) \ge |\mathrm{dist}(b, A) - \mathrm{dist}(b, B)| = \mathrm{dist}(b, A).$$

Similarly, for $a \in A$, $h(A, B) \ge \mathrm{dist}(a, B)$. Therefore $h(A, B) \ge \hat{h}(A, B)$.

For any $x \in X$,

$$\begin{aligned} \mathrm{dist}(x, A) - \mathrm{dist}(x, B) &= \mathrm{dist}(x, A) - \inf_{b\in B} d(x, b) \\ &= \sup_{b\in B} (\mathrm{dist}(x, A) - d(x, b)) \\ &= \sup_{b\in B} \left(\inf_{a\in A} d(x, a) - d(x, b)\right) \\ &= \inf_{a\in A} \sup_{b\in B} (d(x, a) - d(x, b)) \\ &\le \inf_{a\in A} \sup_{b\in B} d(a, b) \\ &= \sup_{b\in B} \mathrm{dist}(b, A). \end{aligned}$$

Similarly $\mathrm{dist}(x, A) - \mathrm{dist}(x, B) \le \sup_{a\in A} \mathrm{dist}(a, B)$. Therefore

$$|\mathrm{dist}(x, A) - \mathrm{dist}(x, B)| \le \max\left\{\sup_{b\in B} \mathrm{dist}(b, A), \sup_{a\in A} \mathrm{dist}(a, B)\right\}.$$

Taking supremum over x gives $h(A, B) \le \hat{h}(A, B)$.

Problem 2.

a) Let $0 < r < 1$ and consider the series

$$\sum_{n=-\infty}^{\infty} r^{|n|} e^{inx}.$$

Show that this series converges uniformly for all $x \in \mathbb{R}$, and that its sum equals

$$P_r(x) = \frac{1 - r^2}{1 - 2r\cos(x) + r^2}.$$

Possible answer: The tails of the series are bounded by a convergent geometric series, independently of x, hence the convergence is uniform. Regarding the sum, we write

$$\begin{aligned}\sum_{n=-\infty}^{\infty} r^{|n|}e^{inx} &= \sum_{n=-1}^{-\infty} r^{|n|}e^{inx} + 1 + \sum_{n=1}^{\infty} r^{|n|}e^{inx} \\ &= \frac{re^{-ix}}{1-re^{-ix}} + 1 + \frac{re^{ix}}{1-re^{ix}} \\ &= \frac{1-r^2}{1-2r\cos(x)+r^2}.\end{aligned}$$

b) Show that

$$P_r(x) \geq 0 \quad \text{for all } x. \tag{1}$$

Possible answer: We have that $\cos(x) \geq -1$, hence $1 - 2r\cos(x) + r^2 \leq (1+r)^2$ and $P_r(x) \geq (1-r)/(1+r) \geq 0$.

c) Show that for every $\delta > 0$, $\delta < \pi$, $P_r(x) \to 0$ uniformly on the intervals $[-\pi, -\delta] \cup [\delta, \pi]$, as $r \uparrow 1$.

Possible answer: For $|x| \geq \delta$, $\cos(x) \leq \cos(\delta)$, and

$$P_r(x) \leq \frac{1-r^2}{1-2r\cos(\delta)+r^2} \quad \to \quad 0, \quad \text{as } r \uparrow 1.$$

d) Show that

$$\int_{-\pi}^{\pi} P_r(x)\,dx = 2\pi. \tag{2}$$

Possible answer: Since the Fourier series defining P_r converges uniformly, we can find the integral by integrating each term. These are all zero except for the term with $n = 0$, which gives the integral 2π.

e) Let f be a continuous 2π periodic function. Show that

$$\lim_{r\uparrow 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)P_r(y)\,dy = f(x).$$

Possible answer: The function f is continuous on $[-\pi, \pi]$, hence it is uniformly continuous here, and given ε, we can find δ such that $|f(x-y) - f(x)| \leq \varepsilon/2$ if $|y| \leq \delta$. Then we have

$$\begin{aligned}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)P_r(y)\,dy - f(x)\right| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x-y)-f(x)\right)P_r(y)\,dy\right| \quad \text{by (2)}, \\ &\leq \frac{1}{2\pi}\int_{|y|>\delta} |f(x-y)-f(x)|\,P_r(y)\,dy \\ &\quad + \frac{1}{2\pi}\int_{-\delta}^{\delta} |f(x-y)-f(x)|\,P_r(y)\,dy \\ &\leq \frac{1}{2\pi}\int_{|y|>\delta} |f(x-y)-f(x)|\,P_r(y)\,dy + \frac{\varepsilon}{2}.\end{aligned}$$

We have that f is bounded, say by M, and next we choose r so close to 1 that $P_r(y) \leq \varepsilon/(4M)$ for $|y| \geq \delta$. Then

$$\frac{1}{2\pi}\int_{|y|>\delta} |f(x-y) - f(x)|\, P_r(y)\, dy \leq \frac{1}{2\pi}\,\frac{\varepsilon}{4M}\, 2M\, 2\pi = \frac{\varepsilon}{2}.$$

f) Assume that f has Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx},$$

show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)P_r(y)\, dy = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{inx},$$

and that the series converges absolutely and uniformly. (**Hint:** Show that the function on the left is differentiable in x)

Possible answer: Let $g(x)$ denote the function $x \mapsto (1/2\pi)\int f(x-y)P_r(y)\, dy$, then a change of variables $z = x - y$ reveals that

$$g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z)P_r(x-z)\, dx,$$

and since P_r is differentiable,

$$g'(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z)P_r'(x-z)\, dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)P_r'(y)\, dy.$$

Therefore the Fourier series of g will converge to $g(x)$ for all x. We now find

$$\begin{aligned}
\langle g, e_n\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}\, dx \\
&= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-y)P_r(y)e^{-inx}\, dydx, \quad z = x - y, \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(z)e^{-inz}\, dz\Big)P_r(y)e^{-iny}\, dy \\
&= c_n r^{|n|}.
\end{aligned}$$

Therefore the Fourier coefficients of g are $c_n r^{|n|}$, and we know that the Fourier series converges to $g(x)$.

g) Show that

$$\lim_{r\uparrow 1} \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{inx} = f(x),$$

uniformly in x.

Possible answer: This now follows from **c)** and **d)**