MAT 2400: Mandatory assignment 1, S-15'

***Deadline**: You must turn in your paper before 2.30 p.m., Thursday, March 5., 2015, in the designated area on the 7th floor of NHA. Remember to use the official front page available on the 7th floor and at*

http://www.mn.uio.no/math/studier/admin/obligatorisk-innlevering/obligforside.pdf

If you due to illness or other circumstances want to extend the deadline, you must apply for an extension to studieinfo@math.uio.no Remember that illness has to be documented by a medical doctor! See

http://www.mn.uio.no/math/studier/admin/obligatorisk-innlevering/index.html

for more information about the rules for mandatory assignments.

***Instructions**: The assignment is compulsory, and students who do not get their paper accepted, will not get access to the final exam. To get the assignment accepted, you need a score of at least 60%. In the evaluation, credit will be given for a clear and well-organized presentation. All questions (points 1a), 1b) etc.) have equal weight. Students who do not get their original paper accepted, but who have made serious and documented attempts to solve the problems, will get one chance of turning in an improved version.*
In solving the problems you may collaborate with others and use tools of all kinds. However, the paper you turn in should be written by you *(by hand or computer) and should reflect your understanding of the material. If we are not convinced that you understand your own paper, we may ask you to give an oral presentation.*

Problem 1. Show that a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ must satisfy $f(n) \geq n$ for all $n \in \mathbb{N}$.

Problem 2. Let (X, d) be a complete metric space. Let $B(x, r)$ denote the open ball centered at $x \in X$ with radius r, i.e.,

$$B(x,r) = \{y \in X \mid d(x,y) < r\},$$

and $\bar{B}(x, r)$ the closed ball of radius r, i.e.,

$$\bar{B}(x,r) = \{y \in X \mid d(x,y) \leq r\}.$$

For any set $C \subseteq X$, let $\overline{C}$ denote its closure.

Is it true that for any complete metric space X,

$$\overline{B(x,r)} = \bar{B}(x,r)\,?$$

Prove this, or give a counterexample.

Problem 3. Let ℓ be the set of sequences of real numbers where only a finite number of terms are different from zero,

$$\ell = \left\{\{x_n\}_{n=1}^{\infty} \mid x_i = 0 \text{ for all but a finite number of } i\text{'s}\right\}.$$

For $x = \{x_n\}$ and $y = \{y_n\}$ in ℓ, define

$$d(x,y) = \sup_{n\in\mathbb{N}} |x_n - y_n|\,.$$

a) Show that d defines a metric on ℓ.

b) Let $u_k \in \ell$ be defined by

$$u_k = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k}, 0, 0, 0, \dots\right\}.$$

Show that $\{u_k\}_{k=1}^{\infty}$ is a Cauchy sequence in (ℓ, d).

c) Show that $\{u_k\}$ is convergent, or show that $\{u_k\}$ is not convergent.

d) Let c_0 be the space of sequences of real numbers, whose limit are 0, i.e.,

$$c_0 = \left\{\{x_n\}_{n=1}^{\infty} \;\middle|\; \lim_{n\to\infty} x_n = 0\right\}.$$

Show that (c_0, d) is a complete metric space (d is the metric above).

e) Prove that ℓ is dense in c_0. What is the limit of u_k?

Problem 4. Let X denote the open interval $(0, \infty) \subset \mathbb{R}$. Let $d : X \times X \to \mathbb{R}$ be defined as

$$d(x, y) = |\ln(x) - \ln(y)|.$$

a) Show that d is a metric, and that (X, d) is complete.

b) Let $f : X \to X$ be a differentiable function and assume that there is a constant $0 < k < 1$ such that

$$x\,|f'(x)| \le k f(x), \text{ for all } x \in X.$$

Show that f has a unique fixed point.