Ark3: Solutions

I have written down some very sketchy solutions to some of the exercises. I have only treated the ones given for the friday sessions, and it has been done rather hastly, so forgive me if there are errors. Still, I hope, they will be useful for you.

Problem 1: Given $\epsilon > 0$. Assume that there is a $\delta < 0$ such that $|x^2 - y^2| < \epsilon$ whenever $|x - y| < \delta$. For any $x > 0$ we let $y = x - \delta$. Then $|x - y| = \delta$ and $x^2 - y^y = |x + y|\,|x - y| = (2x + \delta)\delta > 2x\delta$ which we can make as big as we want, *e.g.*, bigger than ϵ, by choosing x big; *e.g.*, $x > \epsilon/2\delta$. □

Problem 2: Given $\epsilon > 0$. Assume that there is a $\delta < 0$ such that $\left|\frac{1}{x} - \frac{1}{y}\right| < \epsilon$ whenever $|x - y| < \delta$. For any $x > 0$ we let $y = x - \delta$. Then $|x - y| = \delta$ and

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|y - x|}{xy} = \frac{\delta}{x\,|x - \delta|} > \frac{\delta}{x^2}$$

which we can make as big as we want, *e.g.*, bigger than ϵ, by choosing x small; *e.g.*, $x < \sqrt{\frac{\delta}{\epsilon}}$. □

Problem 7:

a) Clearly $\lim_{n\to\infty} x/n = 0$ for any x. The limit is not uniform, since if $\epsilon < 0$ and N are given, $|x|/N > \epsilon$ if $|x| > \epsilon/N$.

b) On the closed interval $[a, b]$ the limit is uniform. If $\epsilon > 0$ is given then $|x|/n < \epsilon$ if $n > \max\{|a|, |b|\}/\epsilon$. □

Problem 8: If $0 < x < 1$ then $\lim_{n\to\infty} x^n = 0$ so x^n converges pointwise to zero. The convergence is not uniform, since if $\epsilon > 0$ is given, the inequality $x^n < \epsilon$ is equivalent to $n > \log\epsilon/\log x$ (remember that the log's are negative,) but as x tends to 1 we know that $\log x$ tends to zero. This means that for any n, we have $x^n > \epsilon$ for x close to 1. □

Problem 11: We have to find $\sup\{|x - x^2| : x \in [0, 1]\}$. Let $f(x) = x - x^2$; it's derivative is $f'(x) = 1 - 2x$ which has a zero at $x = 1/2$ which is a max. This gives that the max-value of f is $1/4$. Hence $\rho(f, g) = 1/4$. □

Problem 13: We have to find the max value of $|\sin x - \cos x|$ on $[0, 2\pi]$. Let $f(x) = \sin x - \cos x$. We find $f'(x) = \cos x + \sin x$ which is zero when $\tan x = -1$, *i.e.*,

$x = 3\pi/4$ or $7\pi/4$. In both cases $|\sin x - \cos x| = \sqrt{2}$, which is bigger than the value at the endpoints (which is 1). Hence $\rho(f, g) = \sqrt{2}$.

We still get the same result in $[\pi, 2\pi]$ since one of stationary points is contained there. □

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