Ark2: Exercises for MAT2400 — Complete and compact spaces, continuous functions

Completeness and compactness are two extremely important concepts. Their strength is illustraded by the two main theorems in these sections, the Banach fixed point theorem and the Extreme value theorem. They are both existence theorems (there is a fixed point; there is a point where f attains its max value) and in both cases the results hang on convergence certain sequences

Of course, continuous functions are everywhere!! No need to underline their importance!

The exercises on this sheet covers the paragraphs **12.4** to **12.3** in the book and the sections *2.1* to *2.6* in Tom's notes.

They are the topics for week 6 (at least some of them, there are too many for one week):

Thursday 10/2: № 2, 7, 8, 16, 17, 19, 20.
Friday 9/2: № 1, 5, 6, 9, 11, 12, 15, 23, 24.

Key words: Cauchy sequences, complete metric spaces, Banach's fixed point theorem, compact metric spaces, bounded metric spaces, the Extreme value theorem, totally bounded metric spaces.

Cauchy sequences

PROBLEM 1. Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ be provided with the metric $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ from exercise 8.b) on Ark1.

a) Show that the sequence $\{a_n\}$ with $a_n = n$ is a Cauchy sequence. Decide if it converges or not.

b) Is the sequence $\{1/n\}$ Cauchy?

c) Show that any sequence $\{a_n\}$ i $\mathbb{R}^+$ converges in $\mathbb{R}^+$ in the metric d above if and only if it converges in $\mathbb{R}^+$ in the standard metric $|x - y|$, and that the limits in the two cases are equal.

PROBLEM 2. Assume that we are given two metrics d_1 and d_2 on a set X. Assume further that there exists a positiv function ϕ defined on the interval $[0, t]$ for some t such that

$$B_{d_1}(x; \phi(r)) \subseteq B_{d_2}(x; r)$$

for all points $x \in X$ and all $r \in [0, t]$.

a) Show that if a sequence $\{a_n\}$ of points in X is Cauchy with respect to d_1 then it is Cauchy with respect to d_2.

b) Show that if a subset $U \subseteq X$ is open with respect to d_2 then it is open with respect to d_1.

c) If the two metrics d_1 and d_2 are equivalent in the sense of *Tom's notes 2.2, Problem 7 (page 27)* they satisfy the condition above, both for d_1 and d_2 and with their roles reversed.

d) Give an example of two metrics on the same set X satisfying the condition above, but not being equivalent. HINT: Take a look at the metrics in problem 23.a) on Ark1.

Continuous functions

PROBLEM 3. *(Tom's notes 4, Problem 2.2 (page 27)).* Let (X, d) be a metric space and let $a \in X$ be a point. Show that $d(x, a)$ is a continuous function of x. (We are using the standard metric $|x - y|$ on $\mathbb{R}$.)

PROBLEM 4. If d_1 and d_2 are metrics on X satisfying the condition in problem 2, and $f : X \to Y$ is a function into a metric space (Y, e), show that if f is continuous with resprect to d_2 then it is continuous with resprect to d_1.

PROBLEM 5. Let $f(x) = \sin x$. For which open intervals $I = \langle a, b\rangle \subseteq [0, 2\pi]$ is $f(I)$ closed? For which I is $f(I)$ open?

PROBLEM 6. Let $a \in [0, 1]$ be a point. Define a function $E_a : C([0, 1]) \to \mathbb{R}$ by sending a function $f \in C([0, 1])$ to its value at a; *i.e.,* it is given by $E_a(f) = f(a)$.

a) Show that E_a is continuous in the sup-norm metric on $C([0, 1])$.

b) Is it continuous in the metric given by $d(f, g) = \int_0^1 |f(x) - g(x)|\, dx$?

PROBLEM 7. *(Cartesian products).* Let (X, d_X) and (Y, d_Y) be two metric spaces. There are several ways of putting a metric on the Cartesian product $X \times Y$. One is using the Manhattan idea; that is, we define d_{MH} by

$$d_{MH}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

a) *(Tom's notes 2.1, Problem 9 (page 24)).* Show that this is a metric.

b) Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be given by $p_X(x, y) = x$ and $p_Y(x, y) = y$. We call them the *projections* onto X and Y. Show that the two projections both are continuous. If (Z, d_Z) is a third metric space, show that a function $f\colon Z \to X \times Y$ is continuous at $z \in Z$ if and only if the two compositions $p_X \circ f$ and $p_Y \circ f$ are.

c) Let $(X', d_{X'})$ and $(Y', d_{Y'})$ be another pair of metric spaces. If $f\colon X \to X'$ and $g\colon Y \to Y'$ are two functions continuous at $x \in X$ and $y \in Y$ repectively, show that then the function $f \times g\colon X \times Y \to X' \times Y'$ given by $(f \times g)(x, y) = (f(x), g(y))$ is continuous at (x, y), where the two Cartesian products are both equipped with their Manhattan metrics. HINT: $p_X \circ f \times g = f \circ p_{X'}$ and $p_Y \circ f \times g = g \circ p_{Y'}$.

d) Show that the diagonal map $\Delta\colon X \to X \times X$ given by $x \mapsto (x, x)$ is continuous (where we still use the Manhattan metric on the product).

e) If f and g are real valued functions on the metric space (X, d), show that the product fg and the sum $f + g$ are continuous, and that fg^{-1} is continuous at points where $g(x) \neq 0$. HINT: Use a) above and the fact that the composition of continuous functions is continuous. You may use the well known facts that the functions xy, $x + y$ and x/y of two real variables are continuous in the Euclidian metric where defined, and swich to the Manhattan metric by problem 4.

f) Show that $|f|$ and the two functions $s(x) = \sup\{f(x), g(x)\}$ and $i(x) = inf\{f(x), g(x)\}$ are continuous. HINT: Use c) above and some appropriate functions of two real variables.

PROBLEM 8. *(Basically Tom's notes 2.6, Problem 13 (page 43)).* Let A and B be two subsets of a metric space (X, d). Define the *distance* between A and B as

$$\operatorname{dist}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

Show that if A and B are compact and disjoint, then $\operatorname{dist}(A, B) > 0$. Give an example of sets A and B with A compact and B closed such that $\operatorname{dist}(A, B) = 0$.

PROBLEM 9. We denote by $\mathscr{T} \subseteq C([-\pi, \pi])$ the subset whose elements are of the form

$$T(x) = \sum_{n=0}^{N} a_n \sin nx + b_n \cos mx,$$

where a_n and b_n are real numbers and we furnish it with the metric induced from the sup-norm metric on $C([-\pi, \pi])$. We call such functions *trigonometric polynomials* or *finite Fourier series.*

Show that the function $D\colon \mathscr{T} \to \mathscr{T}$ sending a trigonometric polynomial $T(x)$ to its derivative $T'(x)$ is not continuous. HINT: For example, take a look at the sequence $\frac{1}{n} \sin nx$.

PROBLEM 10. Let X be a compact metric space and let $f(x)$ and $g(x)$ be two continuous real valued functions on X such that $f(x) < g(x)$. Assume that $0 < g(x)$ for all $x \in X$. Prove that there is a real constant $c < 1$ such that $f(x) \leq cg(x)$ for all $x \in X$.

Complete spaces

PROBLEM 11. Which of the following subsets of $\mathbb{R}^2$ with standard metric are complete?

a) $\{(x, y) : x \geq 0, y \geq 0\}$.
b) $\{(x, y) : x^2 + y^2 > 0\}$.
c) $\mathbb{R}^2 \setminus \mathbb{Q} \times \mathbb{Q}$.

PROBLEM 12. Let (X, d) be a metric space. Let A and B be two subsets of X which we furnish with the induced metrics. Show that if both A and B are complete, then both the intersection $A \cap B$ and the union $A \cup B$ are complete.

PROBLEM 13. *(Tom's notes 2.5, Problem 3 (page 37)).* If A is a subset of a metric space (X, d), the *diameter* $\text{diam}(A)$ is defined to be

$$\text{diam}(A) = \sup\{d(x, y) : x, y \in A\}.$$

Let A_n be a collection of subsets of X such that $A_{n+1} \subseteq A_n$ and $\text{diam}(A_n) \to 0$ as $n \to \infty$. Let $\{a_n\}$ be a sequence of points in X with $a_n \in A_n$. If X is complete, show that a_n converges.

PROBLEM 14. *(Tom's notes 2.5, Problem 4 (page 37)).* Assume that X has two equivalent metrics d and d'. Show that X is complete with respect to d if and only if it is complete with respect to d'.

Compact spaces

PROBLEM 15. We let n be a naturtal number and equip $\mathbb{R}^n$ with the standard metric.
a) Show that the set

$$\Delta_n = \{(x_1, \ldots, x_n) : 0 \leq x_i \text{ for } 1 \leq i \leq n \text{ and } \sum_{i=1}^{n} x_i = 1\}$$

is compact.
b) What about the set

$$\{(x_1, x_2, x_3, x_4, x_5) : 0 \leq x_i \text{ for } 1 \leq i \leq 5 \text{ and } \sum_{i=1}^{4} x_i \leq 1\}?$$

PROBLEM 16. *(Tom's notes 2.5, Problem 10 (page 42)).* Show that a finite union of compact subsets of a metric space X is compact. Show that the intersection of any collection (finite or not) of compact subsets is compact.

PROBLEM 17. *(Tom's notes 2.5, Problem 11 (page 42)).* If (X, d) is a metric space and $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$ is a descending chain of compact subsets, prove that the intersection $\bigcap_i K_i$ is non-empty.

PROBLEM 18. Let (X, d) and (Y, e) be two *compact* metric spaces. Show that the Cartesian product $X \times Y$ is compact when equipped with the Manhattan metric from exercise 7.

PROBLEM 19. Let X be a compact metric space. Let $\mathscr{K}$ be a collection of compact subsets of X with the property that the intersection of any *finite* subcollection of $\mathscr{K}$ is nonempty — *i.e.*, $\bigcap_{i=1}^{r} K_i \neq \emptyset$ whenever $K_1, \dots, K_r \in \mathscr{K}$) — then $\bigcap_{K \in \mathscr{K}} K \neq \emptyset$.

HINT: Assume that $\bigcap_{K \in \mathscr{K}} K = \emptyset$. Then $\{K^c : K \in \mathscr{K}\}$ is an open covering of X with no finite subcovering.

PROBLEM 20. Let $e(x, y)$ be the metric on $\mathbb{R}$ given by $e(x, y) = \frac{|x-y|}{1+|x-y|}$.

a) Show that a sequence $\{a_n\}$ is a Cauchy sequence with respect to e if and only if it is a Cauchy sequence with respect to the standard metric $|x - y|$. HINT: Small balls in the metrics $e(x, y)$ and $|x - y|$ are the same. Take a look at exercise 23.a) from Ark1.

b) Is $\mathbb{R}$ complete with resprect to the metric e? HINT: Exercise 2.a).

c) Show that $\mathbb{R}$ is bounded but not compact with respect to the metric e.

d) Explain (directly) why it is not totally bounded. HINT: Small balls in the metrics $e(x, y)$ and $|x - y|$ are the same.

PROBLEM 21. *(The mole metric).* Let f be the function defined on $\mathbb{R}^+$ by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x > 1. \end{cases}$$

Let $d_m(x, y) = f(|x - y|)$.

a) Show that this a metric on $\mathbb{R}$. We may call it the *mole* metric: If points are close (closer than one meter), their distance is the usual one, but are they far apart (more than one meter) we do not distinguish between their distances; they are just far apart.

b) Show that the "$\mathbb{R}$ of the moles" (*i.e.,* $\mathbb{R}$ with the metric d_m) is complete and bounded but not compact. Is it totally bounded? Why not?

PROBLEM 22. Let n be a natural number and let $P_n([0,1])$ be the subset of $C([0,1])$ consisting of polynomials of degree at most n.

a) Show that the map sending a point $(a_0, .., a_n) \in \mathbb{R}^{n+1}$ to the polynomial $\sum_{i=0}^{n} a_i T^i$ is a continuous, bijective map between $\mathbb{R}^n$ with the Manhattan metric (and hence with the standard, Euclidian metric) and $P_n([0,1])$ with the metric induced from the sup-norm metric.

b) Show that the set of polynomials on $[0,1]$ with degree bounded by n and coefficient bounded by a positive number M (*i.e.,* the poly's $\sum_{i=0}^{n} a_i T^i$ with $|a_i| \leq M$) form a compact set.

c) Show by examples that the set of poly's in *b*) is no longer compact if we let go of either of the boundedness conditions.

Banach's fixed point theorem

PROBLEM 23. Let $f(x) = x^2 + 1/9$. Show that f defines a contractrion $f: [-1/3, 1/3] \to [-1/3, 1/3]$. Determine the fixed point.

PROBLEM 24. Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $F(x,y) = (\frac{1}{3}x + 1, \frac{1}{5}y + 1)$.

a) Show that F is a contraction.

b) Determine the fixed point of F.

PROBLEM 25. *(Tom's notes 2.5, Problem 5 (page 37)).* Assume that $f: [0,1] \to [0,1]$ is a differentiable function, and that there is a number $s < 1$ such that $|f'(x)| < s$ for all $x \in (0,1)$. Show that there is exactly one point $a \in [0,1]$ such that $f(a) = a$.

PROBLEM 26. *(Tom's notes 2.5, Problem 7 (page 37)).* Assume that (X, d) is a complete metric space, and that $f: X \to X$ is a function such that $f^{\circ n}$ is a contraction for some $n \in \mathbb{N}$. Show that f has a unique fixed point.

PROBLEM 27. Let $A: \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. We assume that $\mathbb{R}^n$ has a basis of orthogonal eigenvectors for A, and that all the eigenvalues of A are of absolute value less than one, *i.e.,* if the eigenvalues are $\lambda_1, \ldots, \lambda_n$ then $|\lambda_i| < 1$ for $1 \leq i \leq n$.

a) Show that A is a contraction. What is the fixed point?

b) Let $b \in \mathbb{R}^n$ be any point and let $f(x) = A(x) + b$. Show that f is a contraction. Find an explicit formula for the fixed point of f in terms of A and b. HINT: The linear map $I - A$ is invertible.

PROBLEM 28. Let $0 < \lambda < 1$ be a number, and let $C([0, \lambda])$ be the space of continuous functions on $[0, \lambda]$ with the sup-norm metric.

Define a map $F\colon C([0, \lambda]) \to C([0, \lambda])$ by $F(f) = 1 + \int_0^x f(t)\,dt$.

a) Show that F is a contraction.

b) Start with $f_0(t) = 1$, the constant function with value one, and compute $F(f_0)$, $F^{\circ 2}(f_0)$, and $F^{\circ 3}(f_0)$.

c) Guess a general formula for $F^{\circ n}(f_0)$ and prove it. Find the fixed point for F. (We do not yet know that $C([0, \lambda])$ is complete, and can not apply Banach directly).

d) What happens if you instead start with another function; $\sin x$ for example?

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