Ark4: Exercises for MAT2400 — Differential equations and the Arzelà-Ascoli theorem

The exercises on this sheet covers the sections *3.4* and *3.5* in Tom's notes. They are for the groups on Thursday, February 23 and Friday, February 24. With the following distribution:

Thursday, February 23: No 3, 4, 8, 10, 11, 12.
The rest for Friday

Some Norwegian terminology:
Dense subset = tett undermengde
Separable = separabel
Equicontinuity = ekvikontinuitet

Key words: Differential equations, equicontinuity, dense subsets, separable spaces, the Arzelà-Ascoli theorem.

Differential equations

PROBLEM 1. *(Tom's notes 3.4, Problem 1 (page 61)).*
a) Solve the initial value problem

$$y' = 1 + y^2 \quad y(0) = 0$$

and show that the solution is only defined on the interval $[0, \pi/2)$.
b) Show that the function $1+y^2$ is not uniformly Lipschitz on $[a, b] \times \mathbb{R}$ for any interval $[a, b]$. HINT: Use that $y^2 - z^2 = (y - z)(y + z)$ and let y and z grow towards infinity.

PROBLEM 2. *(Tom's notes 3.4, Problem 2 (page 61)).*
a) Show that the functions

$$y(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq a \\ (t-a)^{\frac{3}{2}} & \text{if } t \geq a \end{cases}$$

where $a \geq 0$ are all solutions of the initial value problem

$$y' = \frac{3}{2} y^{\frac{1}{3}}, \quad y(0) = 0.$$

Remember to check that $y(t)$ is differentiable at $t = a$.

b) Is the function $y^{\frac{1}{3}}$ uniformly Lipschitz on $[0, \infty) \times \mathbb{R}$? Hint: For example, use the mean value theorem to get $y^{\frac{1}{3}} - z^{\frac{1}{3}} = \frac{1}{3}c^{-\frac{2}{3}}(y - z)$ for some number c between y and z. Then let both y and z become small.

Problem 3. *(Linear, homogenous equations).* Let $A = (a_{ij}(t))$ be an $n \times n$ matrix whose entries $a_{ij}(t)$ are continuous functions on a closed interval $[a, b]$.

We equip $\mathbb{R}^n$ with the sup-norm metric given by

$$d(x, y) = \sup\{|x_i - y_j| : i = 1, \dots, n\}$$

for any points $x, y \in \mathbb{R}^n$. As a matter of notation, we put $||y|| = d(y, 0) = \sup\{|y_i| : i = 1, \dots, n\}$.

a) Show that $||Ay|| \leq nM||y||$ where $M = \sup\{|a_{ij}(t)| : t \in [a, b] \text{ and } 1 \leq i, j \leq n\}$, and that Ay is uniformly Lipschitz on $[a, b] \times \mathbb{R}^n$ with constant nM.

b) Let y_0 be in $\mathbb{R}^n$. Show that the initial value problem

$$y' = Ay, \quad y(0) = y_0$$

has a unique solution.

Problem 4. *(Second order diff. eq.Tom's notes 3.4, Problem 3 (page 61)).* Assume we are given a second order initial value problem

$$u''(t) = g(t, u'(t), u(t)) \quad u(0) = a, u'(0) = 0, \tag{$\star$}$$

where g is a given real valued function on $[0, \infty) \times \mathbb{R}^2$. Define $f \colon [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(t, u, v) = \begin{pmatrix} v \\ g(t, u, v) \end{pmatrix}.$$

Show that if $y(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ is a solution of the initial value problem

$$y'(t) = f(t, y(t)) \quad y(0) = \begin{pmatrix} a \\ b \end{pmatrix},$$

then u is a solution of the original problem ($\star$).

Dense subsets and separable spaces

PROBLEM 5. *(Tom's notes 3.5, Problem 1 (page 66)).* Show that $\mathbb{R}^n$ is separable.

PROBLEM 6. *(Tom's notes 2, Problem 3.5 (page 66)).* Show that a subset A of (X, d) is a dense subset if and only if all open balls $B(x; r)$ with $x \in X$ and $r > 0$, contains elements from A.

PROBLEM 7. Show that the set $\{\frac{m}{2^n} : n \in \mathbb{N}, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

The Arzelà-Ascoli theorem

PROBLEM 8. *(Basically Tom's notes 3.5, Problem 4 (page 67)).* Let K be a compact subset of $\mathbb{R}^n$ and let $\{f_n\}$ be a sequence of contractions of K.

a) Show that $\{f_n\}$ has a uniformely convergent subsequence.

b) Let f be the limit of the subsequence above. Show that f has a fixed point. HINT: Lemma 3.5.7 in Tom's can be useful.

c) Find an example of a uniformely convergent sequence of contraction such the limit is *not* a contraction. HINT: For example take $f_n(x) = (1 - 1/n)x$ and $K = [0, 1]$.

PROBLEM 9. *(Tom's notes 3.5, Problem 5 (page 67)).* Recall that a function $f : [-1, 1] \to \mathbb{R}$ is called *Lipschitz continuous with Lipschitz constant K* if

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in [-1, 1]$. Let $\mathscr{K}$ be the subset of $C([-1, 1], \mathbb{R})$ whose members are all Lipschitz continuous functions with Lipschitz constant K such that $f(0) = 0$. Show that $\mathscr{K}$ is compact.

PROBLEM 10. *(Tom's notes 3.5, Problem 7 (page 67)).* A metric space (X, d) is called *locally compact* if for each point $a \in X$, there is a closed ball $\bar{B}(a; r)$ centered at a which is compact. (Recall that $\bar{B}(a; r) = \{x \in X : d(x, a) \leq r\}$.) Show that $\mathbb{R}^n$ is locally compact, but that $C([0, 1], \mathbb{R})$ is not.

PROBLEM 11. Let $I = [-\pi, \pi]$ and $J = [-1, 1]$. Let $\mathscr{S} \subseteq C(I, J)$ be given as $\mathscr{S} = \{\sin nx : n \in \mathbb{N}\}$.

a) Show that for any $x, y \in I$ and any $n \in \mathbb{N}$ there is a c between x and y, such that

$$\sin nx - \sin ny = n \cos nc \, (x - y) \qquad (\star)$$

HINT: Use the mean value theorem.

b) Show that $\mathscr{S}$ is not equicontinuous. HINT: Use ($\star$) above, and work with x and y close to 0, for example closer than $1/n^2$, to see that the right side of ($\star$) can be made as big as you want, even if $x - y$ is a arbitrary small, but fixed, quantity.

c) Is $C(I, J)$ equicontinuous?

d) If both X and Y are compact metric spaces, is it true that $C(X, Y)$ is compact?

PROBLEM 12. Let a parametrised curve $f \colon I \to \mathbb{R}^2$ be given, where $I = [a, b]$; and let $f(t) = (f_1(t), f_2(t))$. We say the curve is of speed bounded by a positive constant M if f_1 and f_2 both are continuously differentiable (so that we untroubled can speek about speed), and if the two derivatives satisfy $|f_i'(t)| \leq M$ for all $t \in I$.

Show that if $\{f_n\}$ is a sequence of curves of speed bounded by M connecting two points A and B in $\mathbb{R}^2$, *i.e.,* that satisfy $f(a) = A$ and $f(b) = B$, then $\{f_n\}$ has a subsequence that converges uniformely on I. HINT: Use the mean value theorem and Arzelà-Ascoli.

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