Ark7: Exercises for MAT2400 — Normes spaces and inner products

The exercises on this sheet cover the sections *4.5* and *4.6* of Tom's notes. They are ment for the groups on Thursday, Mars 15 and Friday, Mars 17. With the following distribution:

Thursday, Mars 15: No 3, 4, 5, 6, 8, 9, 10, 11, 16
The rest for Friday.

Key words: Normed spaces, Inner products, Abel's theorem.

More on Abels's theorem

Problem 1. The aim of this exercise is to show Dirichlet's test for convergence:

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of numbers. Assume that the partial sums $s_N = \sum_{k=0}^{N} a_k$ form a bounded sequence, *i.e.,* there is a constant M such that $\left|\sum_{k=0}^{N} a_k\right| < M$ for all N. Assume further that $\{b_k\}$ decreases monotonically to zero, *i.e.,* $\lim_{k\to\infty} b_k = 0$ and $0 \leq b_{k+1} \leq b_k$. Then the series $\sum_{k=0}^{\infty} a_k b_k$ converges.

a) Use Abel's result on partial summation to show that

$$\sum_{k=M+1}^{N} a_k b_k = s_N b_N - s_M b_M + \sum_{k=M}^{N-1} s_k (b_k - b_{k+1}),$$

where $N > M$ are two integers.

b) Show that

$$\left|\sum_{k=M+1}^{N} a_k b_k\right| \leq 2M b_M.$$

Hint: Use that $b_k - b_{k+1} \geq 0$ and the "telescoping" property of $\sum_{k=M}^{N-1}(b_k - b_{k+1})$.

c) Show that the partial sums $\sum_{k=0}^{N} a_k b_k$ form a Cauchy sequence and hence the series $\sum_{k=0}^{\infty} a_k b_k$ converges.

d) Show by an example that Dirichlet's test is false if we skip the monotonicity condition, that is if we only assume that $\lim_{k\to\infty} b_k = 0$; *i.e.,* find sequences $\{a_k\}$ and $\{b_k\}$ with $\sum_{k=0}^{\infty} a_k$ convergent and with $\lim_{k\to\infty} b_k = 0$ but such that $\sum_{k=0}^{\infty} a_k b_k$ does not converge.

Problem 2. In this exercise we will show by using Dirichlet's test that the series $\sum_{k=0}^{\infty} \frac{\sin kx}{k^a}$ converges for any $a > 0$.

a) Show that

$$2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta).$$

b) Show that

$$2 \sin kx \sin \frac{x}{2} = \cos{(k - \frac{1}{2})x} - \cos{(k + \frac{1}{2})x}.$$

c) Show that

$$\sum_{k=1}^{n} 2 \sin kx \sin \frac{x}{2} = \cos \frac{x}{2} - \cos{(n + \frac{1}{2})x},$$

and hence that $|\sum_{k=1}^{n} \sin kx| \leq \frac{1}{\sin \frac{x}{2}}$ as long as $\sin \frac{x}{2} \neq 0$.

d) Show that $\sum_{k=1}^{\infty} \frac{\sin kx}{k^a}$ converges for any $a > 0$.

Normed spaces

Problem 3. Let $I = [a, b]$ be a closed interval. Show that the following are norms on $C(I, \mathbb{K})$ (where as usual $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$):

a)

$$\|f\|_{\infty} = \sup\{|f(x)| : x \in I\}.$$

b)

$$\|f\|_{1} = \int_{a}^{b} |f(x)| \, dx.$$

Problem 4. *(Tom's notes 4.5, Problem 3 (page 102))*. Let V be a normed space over $\mathbb{K}$. Assume that $\{u_n\}$ and $\{v_n\}$ are sequences in V converging to u and v respectively, and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $\mathbb{K}$ converging to a and β, respectively.

a) Show that $\{u_n + v_n\}$ converges to $u + v$.

b) Show that $\alpha_n u_n$ converges to αu.

c) Show that $\{\alpha_n u_n + \beta_n v_n\}$ converges to $\alpha u + \beta v$.

Problem 5. Let $C^1(I; \mathbb{R})$ be the space of functions on the interval $I = [a, b]$ which are continuously differentiable on I. Show that the following gives a norm on $C^1(I; \mathbb{R})$:

$$\|f\| = \sup\{|f(x)| : x \in I\} + \sup\{|f'(x)| : x \in I\}.$$

PROBLEM 6. *(Tom's notes 4.5, Problem 4 (page 103)).* Let V be a normed space over $\mathbb{K}$.

a) Show the inverse triangle inequality:

$$\big| \|u\| - \|v\| \big| \leq \|u - v\| \text{ for all } u, v \in V.$$

b) If $\{u_n\}$ is a sequence from V converging to u, show that $\{\|u_n\|\}$ converges to $\|u\|$; *i.e.,* the norm is continuous in the metric it defines!

PROBLEM 7. *(Tom's notes 4.5, Problem 10 (page 103)).* We denote by l_1 the set of all sequences $x = \{x_n\}_{n \in \mathbb{N}}$ of real numbers such that $\sum_{n=1}^{\infty} |x_n|$ converges.

a) Check that l_1 is a vector space over $\mathbb{R}$.

b) Show that

$$\|x\| = \sum_{n=1}^{\infty} |x_n|$$

defines a norm on l_1.

c) Show that $e_1, e_2, \ldots, e_n, \ldots$ form a basis for l_1 where e_n is the sequence with $e_n = \{0, 0, 0, \ldots 0, 1, 0, \ldots\}$; *i.e.,* it is zero for all i except for $i = n$ where it is 1.

d) Show that l_1 is complete.

Inner products

PROBLEM 8. *(Tom's notes 4.6, Problem 6 (page 112)).* Let A be a real, symmetric $n \times n$ matrix all whose eigenvalues are strictly positive. Show that $\langle x, y \rangle = x^t A y$ defines an inner product on $\mathbb{R}^n$.

PROBLEM 9. *(Tom's notes 4.6, Problem 8 (page 112)).* Assume that $\{u_n\}$ and $\{v_n\}$ are two sequences from an inner product space converging to u and v respectively. Show that $\langle u_n, v_n \rangle$ tends to $\langle u, v \rangle$ as $n \to \infty$.

PROBLEM 10. *(Tom's notes 4.6, Problem 9 (page 113)).* Show that if the norm $\|\cdot\|$ is defined from an inner product by $\|u\| = \langle u, u \rangle^{\frac{1}{2}}$, then we have the parallellogram law:

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

where u and v are any elements from V. Show that neither the norm on $\mathbb{R}^2$ given by $\|(x, y)\| = \max\{|x|, |y|\}$ nor the one given by $\|(x, y)\| = |x| + |y|$ comes from an inner product.

PROBLEM 11. Prove the Cauchy-Schwartz inequality for sequences $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ of real numbers:

$$(\sum_{k=1}^{n} x_k y_k)^2 \leq (\sum_{k=1}^{n} {x_k}^2)(\sum_{k=1}^{n} {y_k}^2).$$

HINT: You may use proposition 4.6.4 in Tom's notes.

PROBLEM 12. *(Basically Tom's notes 4.6, Problem 13 (page 113)).* Let l_2 be the set of sequences $\{x_k\}$ of real numbers such that $\sum_{k=1}^{\infty} x_k^2$ converges.

a) Prove the Cauchy-Schwartz inequality for l_2, *i.e.*, show that if $x = \{x_k\}$ and $y = \{y_k\}$ are from l_2, then

$$(\sum_{k=1}^{\infty} x_k y_k)^2 \leq (\sum_{k=1}^{\infty} {x_k}^2)(\sum_{k=1}^{\infty} {y_k}^2).$$

HINT: Use problem 11 to bound partial sums.

b) Show that l_2 is a real vector space. HINT: Use 12.a) to see that l_2 is closed under addition.

c) Show that $\langle x, y \rangle = \sum_{k=1}^{\infty} x_n y_n$ defines an inner product on l_2.

d) Show that l_2 is complete.

e) For n a natural number, let e_n be the sequence with all components equalt to zero except the n-th one, which is 1. Show that $e_1, e_2, e_3, \dots, e_n, ..$ form an orthonormal basis for l_2. Why is this not in conflict with exercise 7 ?

PROBLEM 13. *(Tom's notes 4.6, Problem 13, d) (page 113)).* Let V be an inner product space with an orthonormal basis $\{v_1, v_2, \dots, v_n, \dots\}$. Assume that for any square summable sequnce $\{\alpha_n\}$ — *i.e.*, a sequence from l_2 — there is an element $u \in V$ satisfying $\langle u, v_i \rangle = \alpha_i$ for all $i \in \mathbb{N}$. Show that V is complete.

PROBLEM 14. Let f and g be two continuous complex valued functions on $[a, b]$, let

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\, dx.$$

a) Show that this defines an inner product on the space of continuous functions $C([a, b]; \mathbb{C})$.

b) Show the Cauchy-Schwartz inequality for integrals, *i.e.*, show that

$$\left| \int_a^b f(x)\overline{g(x)}\, dx) \right|^2 \leq \int_a^b |f(x)|^2\, dx \int_a^b |g(x)|^2\, dx.$$

PROBLEM 15. Let $f, g \in C([-1,1];\mathbb{R})$.
a) Show that the improper integral $\int_{-1}^{1} \frac{f(t)g(t)}{\sqrt{1-t^2}}\,dt$ is convergent, and that

$$\langle\, f, g\,\rangle = \int_{-1}^{1} \frac{f(t)g(t)}{\sqrt{1-t^2}}\,dt$$

defines an inner product on $C([-1,1];\mathbb{R})$.
b) Show that the functions $T_n(t) = \cos(n \arccos t)$ form an orthogonal set in $C([-1,1];\mathbb{R})$ with respect to this inner product. HINT: The substitution $t = \cos u$ may be helpful.
c) Show that $T_n(t)$ is a polynomial in t of degree n. (They are called Chebyshev polynomials).

PROBLEM 16. If $S \subseteq V$ is a subset of an inner product space V, we let

$$S^{\perp} = \{v \in V : \langle\, v, s\,\rangle = 0 \text{ for all } s \in S\}.$$

a) Show that $S^{\perp}$ is a closed vector subspace of V.
b) Show that if $S \subseteq T$, then $T^{\perp} \subseteq S^{\perp}$.
c) Show that $(S^{\perp})^{\perp}$ is the smallest closed subspace of V containing S
d) Show that if S is dense in V then $S^{\perp} = 0$. Does the converse hold?s

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