MAT2400 Mandatory assignment 1

All the individual subproblems carry the same weight, and to pass you need a score of at least 50% (which means 6 of the 11 subproblems). The assignment must be handed in in the special box at the 7. floor in Niels Henrik Abels hus (mattebygget) before 14.30 Thursday 8. March. Remember to fill in and attach a frontpage — frontpages are found nearby the box or you may find it on the net.

Problem 1.

Recall that a map between to metric spaces is called an *isometry* if it preserves distances,*i.e.,* if (X, d_X) and (Y, d_Y) are the two metric spaces and $\phi\colon X \to Y$ the map, we have

$$d_X(x, y) = d_Y(\phi(x), \phi(y)).$$

a) With the notation above, assume that ϕ in addition to being an isometry also is invertible, *i.e.,* there is a continuous map $\psi\colon Y \to X$ with $\phi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \phi = \mathrm{id}_X$. Show that (X, d_X) is complete if and only if (Y, d_Y) is complete.

b) Show that

$$d(x, y) = |x - y| + \left|\frac{1}{x} - \frac{1}{y}\right|$$

defines a metric on $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.

c) Let (X, e) be the metric space with underlying set the subset $\{(x, y) : y = \frac{1}{x}, x > 0\}$ of $\mathbb{R}^2$ and with metric induced from the Manhattan-metric $|x_1 - x_2| + |y_1 - y_2|$ on $\mathbb{R}^2$. Show that mapping $\phi\colon X \to \mathbb{R}^+$ given by $\phi(x, y) = x$ is an invertible isometry.

d) Show that the metric from 1.b) is complete.

Problem 2.

In this exercise we work in the space $C(\mathbb{R}^+, \mathbb{R})$ of continuous, *bounded* functions on $\mathbb{R}^+$. The metric is the sup-norm metric $\rho(f, g) = \sup\{|f(x) - g(x)| : x \in \mathbb{R}^+\}$. You can freely use (without proof) that this is well defined and that it is a metric.

Let the following sequence of functions

$$f_n(t) = \sin\sqrt{t + n^2\pi^2}$$

for $n = 1, 2, 3\ldots$ and $t > 0$ be given.

a) Show that the family $\{f_n\}$ is bounded and equicontinuous in the sup-norm metric ρ. HINT: The mean value theorem will be indispensable.

b) Show that $f_n(t)$ tends pointwise to zero, but not uniformly. HINT: Again, use the mean value theorem, and a juicy choice of some values for t.

c) Does the analogue of the Arzelà-Ascoli theorem hold in $C(\mathbb{R}^+, \mathbb{R})$? Comment.

Problem 3.

Let K be a compact metric space with metric d.

a) Show that the product $K \times K$ with the metric

$$d_{MH}((x_1, x_2)(y_1, y_2)) = d(x_1, y_1) + d(x_2, y_2)$$

is compact.

b) Show that there are points $x, y \in K$ such that

$$d(x_0, y_0) = \sup\{d(x, y) : x, y \in K\}.$$

This number is called the *diametre* of K.

Problem 4.

Let $K \subseteq \mathbb{R}^m$ be a compact subset where we use the standard metric on $\mathbb{R}^m$. Let $\mathscr{I} \subseteq C(K, K)$ be the subset $\mathscr{I} = \{f : K \to K : d(f(x), f(y)) = d(x, y), x \text{ and } y \in K\}$; *i.e.,* it is the subset of isometries.

a) Show that $\mathscr{I}$ is closed an bounded.

b) Show that $\mathscr{I}$ is compact. HINT: Use the Arzelà-Ascoli theorem.