MAT2400 Mandatory assignment 2

All the individual subproblems carry the same weight, and to pass you need a score of at least 50% (which means 6 of the 12 subproblems). The assignment must be handed in in the special box at the 7. floor in Niels Henrik Abels hus (mattebygget) before 14.30 Thursday 3. May. Remember to fill in and attach a frontpage — frontpages are found nearby the box or you may find it on the net.

Notation: For any function f of one real variable, $f(a^+)$ denotes the limit of $f(x)$ when x tends to a from above (if it exists); *i.e.*, $f(a^+) = \lim_{t \to a^+} f(t)$. Similarly, $f(a^-)$ denotes the limit of $f(x)$ when x tends to a from below (if it exists).

Problem 1.

The aim of this problem is to study a phenomenon which is called *Gibb's* phenomenon. At every simple jump discontinuity of a function f, the partial sums of the Fourier series of f "overshoots" near the singularity by an amount about 9% of the "jump" of the function.

To be presise, assume that $f(x)$ has a jump singularity at a; *i.e.*, $d = f(a^+) - f(a^-) \neq 0$ and is continuous elsewhere in a neighbourhood of a. For simplicity we assume that $d > 0$. We let $s_n(x)$ be the n-th partial sum of the Fourier series of f. Then there is a sequence $\{x_n\}$ tending to a from above such that $s_n(x_n) > f(a^+) + \alpha d$, where the constant α satisfies $\alpha \approx 0.089$, *i.e.*, about 9%. There is a similar sequence $\{y_n\}$ tending to a from below with $s_n(y_n) < f(a^-) - \alpha d$

In this this problem we will study Gibbs phenomenon for the particular function given in $(-\pi, \pi)$ by:

$$d(x) = \begin{cases} \pi/2 & \text{if } 0 < x < \pi \\ 0 & \text{if } x = 0 \\ -\pi/2 & \text{if } -\pi < x < 0 \end{cases}$$

a) Compute the Fourier coefficients of d, and show that we have the equality

$$d(x) = 2 \sum_{k=1}^{\infty} \frac{\sin{(2k-1)x}}{2k-1}$$

for all $x \in (-\pi, \pi)$.

b) Let the partial sums of the Fourier series of $d(x)$ be denoted by $d_n(x)$. Show that we have

$$d_n(x) = 2\sum_{k=1}^{n} \frac{\sin{(2k-1)x}}{2k-1} = \int_0^x \frac{\sin 2nt}{\sin t}\,dt.$$

HINT: Compute the derivative of $d_n(x)$ and use that $2\sum_{k=1}^{n} \cos(2k-1)x = \frac{\sin 2nx}{\sin x}$. To prove the last formula, use the for us now well used and classical formula $2\sin\alpha\cos\beta = \sin(\beta+\alpha) - \sin(\beta-\alpha)$.

c) Show that for $t \geq 0$ the following inequality holds true

$$0 \leq t - \sin t \leq t^3/6.$$

Use that inequality to prove that

$$\left|\frac{1}{\sin t} - \frac{1}{t}\right| \leq \frac{\pi}{12}t,$$

when $0 < t \leq \pi/2$.

d) Prove that for all n and all $0 < x < \pi/2$:

$$\left|d_n(x) - \int_0^{2nx} \frac{\sin u}{u}\,du\right| < \frac{\pi}{24}x^2$$

and use this to prove that for a given $\epsilon > 0$ there is an n_0 such that if $n \geq n_0$, then

$$d_n(\pi/2n) > \pi/2 + \alpha\pi - \epsilon$$

where the constant α is given by $\alpha = \pi^{-1}(\int_0^\pi \frac{\sin u}{u}\,du - \pi/2)$. Hence

$$d_n(\pi/2n) \geq \pi/2 + 0.089\pi.$$

because one may compute $\alpha = 0.08949\ldots$. (You can consider that value as given!).

Problem 2.

Let $C = C([0,1],\mathbb{R})$ be the Banach space of continuous real valued functions on the interval $[0,1]$ with norm given by $\|f\| = \sup\{|f(x)| : x \in [0,1]\}$. Fix an element $g \in C$, and let $I\colon C \to C$ be the map given by

$$I(f)(x) = \int_0^x f(t)g(t)\,dt.$$

a) Show that I is a bounded linear map; that is, I is linear and there is a positive constant M such that $\|I(f)\| \leq M\|f\|$ for all $f \in C$. Determine the least such constant if g is a positive function.

b) Show that the map $I\colon C \to C$ is uniformly continuous.

c) Show that for any bounded subset $A \subseteq C$ the set $I(A) \subseteq C$ is equicontinuous.

d) Show that the closure $\overline{I(A)}$ is a compact subset.

e) For each real number $\lambda \neq 0$, let $V_\lambda = \{f \in C : I(f) = \lambda f\}$. Show that V_λ is a subvector space of C. Determine all functions in V_λ.

Problem 3.

Let $F(x)$ be a strictly increasing function. For any half open interval $I = (a, b]$ define $m(I) = F(b) - F(a)$, and for any set $E \subseteq \mathbb{R}$, let

$$\nu^*(E) = \inf\{\sum_{I \in \mathcal{A}} m(I) : \mathcal{A}\}$$

where $\mathcal{A}$ runs through all countable coverings of E by half open intervals $(a, b]$.

a) Show that $\nu^*(E) \geq 0$, and that ν^* is monotone; *i.e.*, $\nu^*(E') \leq \nu^*(E)$ whenever $E' \subseteq E$.

b) Show that ν^* is semiadditive; that is

$$\nu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \nu^*(E_n)$$

for any family $\{E_n\}$ of subsets of $\mathbb{R}$.

c) If $x \in \mathbb{R}$, show that $\nu^*(\{x\}) = F(x) - F(x^-)$, and hence $\nu^*\{x\} = 0$ if and only if F is continuous from the left at x.