Trial Exam — MAT2400 spring 2012

PROBLEM 1. In this problem $f_n(x) = \arctan nx$ for each natural number n and a denotes a positive real number; *i.e.*, $a > 0$.

a) Show that each of the functions f_n is uniformly continuous on $(0, \infty)$?

b) Show that the sequence $\{f_n\}$ converges uniformly on (a, ∞). Is the convergence uniform on $[0, \infty)$? HINT: It may be helpful to first show that $\frac{\pi}{2} - \arctan x$ is a decreasing function for $x > 0$.

PROBLEM 2.

a) La $f(x)\colon \mathbb{R} \to \mathbb{R}$ be defined on $[-\pi, \pi]$ by

$$f(x) = \begin{cases} x & \text{when } x \in (-\pi, \pi) \\ 0 & \text{when } x = \pm\pi \end{cases}$$

and then extended by peridiocity. Show that the Fourier series of f is

$$2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

For which numbers x does the Fourier series converge to $f(x)$? Is the convergence uniform on $[-\pi, \pi]$?

b) Use 2. a) to show that

$$x^2/2 = \frac{\pi^2}{6} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

for $x \in (-\pi, \pi)$. What are the values of the sum for $x = \pm\pi$? Is the convergence uniform on $[-\pi, \pi]$?

c) Determine the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

PROBLEM 3.

a) What does it mean that a function $f\colon \mathbb{R} \to \overline{\mathbb{R}}$ is measurable?

b) Let $\{f_n(x)\}_{n \in \mathbb{N}}$ be a sequence of measurable functions. Show that the function $g(x) = \sup_{n \in \mathbb{N}} f(x)$ is measurable.

c) Show that if f and g are two measurable functions, then their sum $f+g$ is measurable.

PROBLEM 4. In this problem a denotes a positive real constant. For every natural number n, we let $U_n = (-a - \frac{1}{n}, a + \frac{1}{n})$, and we let $U_0 = \mathbb{R}$.

a) Show that U_n is a decreasing sequence of open sets whose intersection is the interval $I = [-a, a]$.

b) Let $f(x)$ be the following function:

$$f(x) = \begin{cases} 0 & x \in I \\ \frac{1}{n+1} & x \in \mathbb{Q} \cap (U_n \setminus U_{n+1}), n \geq 0 \\ -\frac{1}{n+1} & x \in \mathbb{Q}^c \cap (U_n \setminus U_{n+1}), n \geq 0. \end{cases}$$

where $\mathbb{Q}$ as usual denortes the set of rational numbers. Explain why f is a well defined function on $\mathbb{R}$, and show that f is continuous at every point in I.

c) Show that f is not continuous at points $x \notin I$.

d) Show that f is measurable. Is f integrable over $[-1, 1]$?