Exam MAT2400 Spring 2012 — Solutions

Problem 1:

a) We want to apply the mean value theorem to $g_n(x)$, and to that end, we compute the derivative of $g_n(x)$. We find

$$g_n'(x) = -\frac{2n}{(1+nx)^3}.$$

The mean value theorem applied to g_n over the interval $[x, y] \subseteq [1, \infty)$ then gives us

$$|g_n(x) - g_n(y)| = \frac{2n}{(1+nc)^3}\,|x-y| \le \frac{2n}{(1+n)^3}\,|x-y| \le \frac{2}{n^2}\,|x-y| \qquad (\divideontimes)$$

where c is a number beteween x and y — in particular $c \ge 1$. So, for any $\epsilon > 0$, if $\delta \le \frac{n^2}{2}\epsilon$, we get from ($\divideontimes$) that

$$|g_n(x) - g_n(y)| \le \frac{2}{n^2}\,|x-y| \le \frac{2}{n^2}\delta \le \epsilon$$

when $|x - y| \le \delta$. And this is true for all x and y in $[1, \infty)$, hence g_n is uniformly continuous.

The functions $g_n(x)$ are all uniformly continuous on $[0, 1]$ since every continuous function on a compact set is uniformly continuous.

b) One has $\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{1}{(1+nx)^2} = 0$ when $x > 0$. Furthermore, since $x \ge 1$,

$$\left|\frac{1}{(1+nx)^2}\right| \le \frac{1}{(1+n)^2} \le \frac{1}{n^2},$$

which shows that the convergence is uniform. Indeed, $\frac{1}{n^2}$ tends to 0 when n tends to ∞, and of course $\frac{1}{n^2}$ is independent of x.

The sequence $\{g_n\}$ does not converge uniformly on $[0, 1]$. The limit is not continuous, being 0 if $x > 0$ and 1 if $x = 0$, whereas all the functions g_n in the sequence are. ❑

Problem 2:

a) The function f is even (*i.e.*, $f(-x) = f(x)$) so its Fourier series is a pure cosine series. We compute the coefficients. If $n > 0$, they are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^a \cos nx\,dx = \frac{2}{\pi}\bigg|_0^a \frac{1}{n}\sin nx = \frac{2}{n\pi}\sin na,$$

and if $n = 0$,

$$a_0 = \frac{1}{2\pi}\int_{-a}^{a} dx = \frac{a}{\pi}.$$

This gives the Fourier series of f:

$$\frac{a}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin na \cos nx}{n}.$$

Or, if you prefere to do it in the complex way:

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-a}^{a} e^{-inx}\,dx = \frac{1}{2\pi(-in)}\bigg|_{-a}^{a} e^{-inx} = \\ &= \frac{1}{n\pi}\cdot\frac{e^{ina}-e^{-ina}}{2i} = \frac{1}{n\pi}\sin na, \end{aligned}$$

where $n \neq 0$. The constant term, c_0, is the same as above, *i.e.*, $c_0 = a_0 = \frac{a}{\pi}$. Grouping the terms corresponding to n amd $-n$ together, and using that $\sin(-na) = -\sin na$, we find that the Fourier series of $f(x)$ is:

$$\frac{a}{\pi} + \sum_{n=-\infty, n\neq 0}^{\infty} c_n e^{inx} = \frac{a}{\pi} + \sum_{n=1}^{\infty}\frac{1}{n\pi}\sin na(e^{inx}+e^{-inx}) = \frac{a}{\pi} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin na\cos nx.$$

b) The function f being picewise constant clearly has one-sided derivatives everywhere, hence converges to $(f(x^+) + f(x^-))/2$ for all x by Dini's test (or more presicely its second corollary in Tom's notes). Setting $x = a$ and using that $2\sin\alpha\cos\alpha = \sin 2\alpha$, we therefore get

$$\frac{1}{2} = \frac{a}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin na\cos na}{n} = \frac{a}{\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2na}{n},$$

and from this we conclude that

$$\frac{\pi}{2} - a = \sum_{n=1}^{\infty}\frac{\sin 2na}{n}.$$

c) Integrating the series in 2a) term by term from 0 to x, we obtain:

$$\frac{ax}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin na\sin nx}{n^2}. \qquad (*)$$

We are in this case allowed to swich the order in which we perform the summation and the integration (justified either by the theorem in Tom's notes which states that this is indeed true for all integrated Fourier series, or by observing that our series (❋) converges uniformly by Weierstrass' M-test), so the series (❋) converges to the following function:

$$g_1(x) = \int_0^x f(t)\,dt = \begin{cases} a & x \in (a, \pi] \\ x & x \in [-a, a] \\ -a & x \in [-\pi, -a), \end{cases}$$

which gives that the series in the problem converges to $g_1(x) - \frac{ax}{\pi}$, *i.e.*, the function

$$g(x) = \begin{cases} a(1 - \frac{x}{\pi}) & x \in (a, \pi] \\ x(1 - \frac{a}{\pi}) & x \in (-a, a) \\ -a(1 + \frac{x}{\pi}) & x \in [-\pi, -a). \end{cases}$$

❑

PROBLEM 3:

a) Let $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. If f is measurable, these two functions are both measurable. We say that f is integrable if it is measurable and both the non-negative functions f^+ and f^- are integrable, that is if both $\int f^+\,d\mu < \infty$ and $\int f^-\,d\mu < \infty$.

b) Let $E = \{x : f(x) = \infty\} = \{x : f^+(x) = \infty\}$. We have $\int_E f^+\,d\mu \leq \int f^+\,d\mu$, and if $\mu(E) > 0$ then $\int_E f^+\,d\mu = \infty \cdot \mu(E) = \infty$, so f is not integrable. The second part of the problem follows in a similar way (*e.g.*, by considering the integrable function $-f$ and noting that $(-f)^+ = f^-$).

c) Aiming for a contradiction, we assume that f is not zero almost everywhere. That means that at leat one of the sets $E = \{x : f(x) > 0\}$ and $E' = \{x : f(x) < 0\}$ are of positive measure (indeed, $\{x : f(x) \neq 0\} = E \cup E'$), and we may without loss of generality assume that $\mu(E) > 0$ (if not, replace f by $-f$).

Let $E_n = \{x : f(x) \geq \frac{1}{n}\}$. Then $\{E_n\}$ is an increasing sequence of measurable sets satisfying $E = \bigcup_{n=1}^{\infty} E_n$. Hence $\mu(E) = \lim_{n\to\infty} \mu(E_n)$, and since $\mu(E) > 0$, we get $\mu(E_n) > 0$ for n big. Then, for such an n, we get $0 < \frac{\mu(E_n)}{n} \leq \int_{E_n} f\,d\mu$ which contradicts the fact that $\int_{E_n} f\,d\mu = 0$. ❑

PROBLEM 4:

a) Let $x \in U_\epsilon$ be given, and let δ be such that if $|x-s| < \delta$ and $|x-t| < \delta$, then $|f(s)-f(t)| < \epsilon$.

It suffices to show that if $|h| < \delta/2$, then $x+h \in U_\epsilon$. To this end, assume that $|x+h-s| < \delta/2$ and $|x+h-t| < \delta/2$. Then

$$|x-s| \leq |x+h-s| + |h| < \delta/2 + \delta/2 = \delta$$

and

$$|x-t| \leq |x+h-t| + |h| < \delta/2 + \delta/2 = \delta.$$

Hence by the definition of U_ϵ, we get $|f(s)-f(t)| < \epsilon$ because $x \in U_\epsilon$. It follows that $x+h \in U_\epsilon$.

b) Let $x \in \bigcap_{n\in\mathbb{N}} U_{\frac{1}{n}}$ and let $\epsilon > 0$ be given. Pick a natural number N such that $\frac{1}{N} < \epsilon$. Since $x \in U_{\frac{1}{N}}$ there is a $\delta > 0$ such that the two inequalities $|x-s| < \delta$ and $|x-t| < \delta$ imply that $|f(s)-f(t)| < \frac{1}{N} < \epsilon$. In particular, we my take $s = x$ and get $|f(x)-f(t)| < \epsilon$ whenever $|x-t| < \delta$. Hence f is continuous at x.

Assume that f is continuous at x. There is $\delta > 0$ with $|f(x)-f(t)| < \epsilon/2$ once $|x-y| < \delta$. Then if $|x-s| < \delta$ and $|x-t| < \delta$ we have

$$|f(s)-f(t)| \leq |f(x)-f(s)| + |f(x)-f(t)| \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

This shows that $x \in U_\epsilon$ for any $\epsilon > 0$, in particular $x \in U_{\frac{1}{n}}$ for all $n \in \mathbb{N}$.

c) Let $x \in A$ and $\epsilon > 0$ be given. There is an N such that $\left|3^{-N}\right| < \epsilon$; then if $y \in U_n$ for $n \geq N$, we have

$$|f(x)-f(y)| = \left|3^{-n(y)}\right| \leq \left|3^{-N}\right| < \epsilon,$$

and we are through since U_n is open.

d) Let $x \notin A$ and assume that f is continuous in x. Let $\epsilon = \frac{1}{2}\cdot 3^{-n(x)}$. By continuity of f there is an open set V containing x such that if $y \in V$ then $|f(x)-f(y)| < \frac{1}{2}\cdot 3^{-n(x)}$.

Then $|f(y)| > |f(x)| - \frac{1}{2}\cdot 3^{-n(x)} = \frac{1}{2}\cdot 3^{-n(x)} > 3^{-(n(x)+1)}$, and it follows from this that $V \cap U_{n(x)} \subseteq U_{n(x)} \setminus U_{n(x)+1}$.

Since both $\mathbb{Q}$ and $\mathbb{Q}^c$ are dense in $\mathbb{R}$, there are points both in $\mathbb{Q} \cap V \cap U_{n(x)}$ and in $\mathbb{Q}^c \cap V \cap U_{n(x)}$.

If $x \in \mathbb{Q}$, pick $z \in \mathbb{Q}^c \cap V \cap U_{n(x)}$. Then $f(z) = -3^{-n(x)}$, and $f(x)-f(z) = 2\cdot 3^{-n(x)}$. If $x \in \mathbb{Q}^c$, a $z \in \mathbb{Q} \cap V \cap U_{n(x)}$ will do the job: $f(x)-f(z) = -2\cdot 3^{-n(x)}$.

In both cases $|f(x)-f(z)| = 2\cdot 3^{-n(x)} > \frac{1}{2}\cdot 3^{-n(x)} = \epsilon$ — a contradiction. ❏

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