Problem 1

a

Assume Y is complete and that x_n is a Cauchy sequence in X. Since ϕ is an isometry, $y_n = \phi(x_n)$ is also a Cauchy sequence. Moreover, since Y is complete, there is then a $y \in Y$ such that $y_n \to y$. Let $x = \psi(y)$ and note that $y = \phi(x)$, so

$$d_X(x_n, x) = d_Y\big(\phi(x_n), \phi(x)\big) = d_Y(y_n, y).$$

As $y_n \to y$ we see that $x_n \to x$. Hence x_n converges, so X is complete.

To prove that Y is complete if X is complete just switch the places of X and Y and ϕ and ψ in the argument above, and note that ψ is an isometry since

$$d_X\big(\psi(y'), \psi(y'')\big) = d_Y\big(\phi \circ \psi(y'), \phi \circ \psi(y'')\big) = d_Y(y', y'').$$

b

We have to check positive definiteness, symmetry and the triangle inequality. For positivity, note that $|x - y| \geq 0$ and $\left|\frac{1}{x} - \frac{1}{y}\right| \geq 0$, so $d(x, y) \geq 0$. If $d(x, y) = 0$ then in particular $|x - y| = 0$, so $x = y$. Hence d is definite.

For symmetry, note that

$$d(x, y) = |x - y| + \left|\frac{1}{x} - \frac{1}{y}\right| = |y - x| + \left|\frac{1}{y} - \frac{1}{x}\right| = d(y, x).$$

For the triangle inequality, given $x, y, z \in \mathbb{R}^+$, note that by the usual triangle inequality we get

$$d(x, z) = |x - z| + \left|\frac{1}{x} - \frac{1}{z}\right| \leq |x - y| + |y - z| + \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} - \frac{1}{z}\right| = d(x, y) + d(y, z).$$

Hence d is a metric.

c

Given $x = \big(x_1, \frac{1}{x_2}\big), y = \big(y_1, \frac{1}{y_1}\big) \in X \subset \mathbb{R}^2$ we get

$$e(x, y) = d_{\mathbb{R}^2}(x, y) = |x_1 - y_1| + \left|\frac{1}{x_1} - \frac{1}{y_1}\right| = d(x_1, y_1) = d\big(\phi(x), \phi(y)\big),$$

whence ϕ is an isometry. Let $\psi \colon \mathbb{R}^+ \to X$ be given by $\psi(x) = \big(x, \frac{1}{x}\big)$. The map ϕ is a bijection since $\phi \circ \psi(x) = x$ for all $x \in \mathbb{R}^+$ and $\psi \circ \phi\big(x, \frac{1}{x}\big) = \big(x, \frac{1}{x}\big)$ for all $\big(x, \frac{1}{x}\big) \in X$. By a similar argument as in (a) ψ is an isometry and hence continuous. Thus ψ is a continuous inverse of ϕ, so ϕ is an invertible isometry.

d

Recall that X is a closed subset of $\mathbb{R}^2$, and that $\mathbb{R}^2$ is complete. Hence X is complete. By (a), since $\phi\colon X \to \mathbb{R}^2$ is an invertible isometry, this implies that $\mathbb{R}^+$ is complete.

Problem 2

a

Note that $|f_n(t)| = |\sin\sqrt{t + n^2\pi^2}| \leq 1$ since $-1 \leq \sin x \leq 1$ for all $x \in \mathbb{R}$. Thus,

$$\rho(f_1, f_n) = \sup_t |f_1(t) - f_n(t)| \leq \sup_t |f_1(t)| + |f_n(t)| \leq 1 + 1 = 2$$

whence $\{f_n\}$ is bounded.

The mean value theorem applied to f_n shows us that

$$|f_n(x) - f_n(y)| = |f_n'(c)| \cdot |x - y|,$$

so we would like to estimate $|f_n'(c)|$. Note that

$$|f_n'(c)| = \left|\frac{\cos\sqrt{c + n^2\pi^2}}{2\sqrt{c + n^2\pi^2}}\right| \leq 1$$

for all n and c.

Hence given any $\epsilon > 0$ we can choose $\delta = \epsilon$ to get

$$|f_n(x) - f_n(y)| \leq |x - y| < \delta = \epsilon$$

whenever $|x - y| < \delta$. Hence $\{f_n\}$ is equicontinuous.

b

Extend f_n to $[0, \infty)$ by continuity, so that $f_n(0) = 0$. For $t \in \mathbb{R}^+$

$$|f_n(t)| = |f_n(t) - f_n(0)| = |f_n'(c)| \cdot |t - 0|.$$

Now consider $f_n'(c)$. We have

$$|f_n'(c)| = \left|\frac{\cos\sqrt{c + n^2\pi^2}}{2\sqrt{c + n^2\pi^2}}\right| \leq \frac{1}{2\sqrt{c + n^2\pi^2}} \leq \frac{1}{2n\pi}.$$

Thus $|f_n(t)| \leq \frac{|t|}{2n\pi}$, so $f_n(t) \to 0$ as $n \to \infty$. Hence f_n converges pointwise to zero.

On the other hand,

$$\begin{aligned}\sup_t |f_n(t)| &\geq |f_n\big((n + 1/4)\pi^2\big)| = |\sin\sqrt{(n + 1/4)\pi^2 + n^2\pi^2}| \\ &= |\sin\sqrt{(n + 1/2)^2\pi^2}| = |\sin(n + 1/2)\pi| = 1,\end{aligned}$$

so $\rho(f_n, 0) \geq 1$. Hence f_n cannot converge uniformly to zero.

c

Consider the set $\{f_n\} \subset C(\mathbb{R}^+, \mathbb{R})$. This set is bounded and equicontinuous. By problem (b) the sequence f_n does not converge uniformly to the zero function, so the set is closed. If an analogue of the Arzelà-Ascoli theorem should hold this would imply that a subsequence of f_n converges. However, such a subsequence could only converge to the zero function, which is impossible. Hence such an analogue cannot hold.

Problem 3

a

Let $z_n = (x_n, y_n)$ be a sequence in $K \times K$. Then x_n is a sequence in K, so has a convergent subsequence, say x_{n_k}, which converges to some $x \in K$. The corresponding subsequence y_{n_k} is again a sequence in K, so has a convergent subsequence, say $y_{n_{k_l}}$, which converges to $y \in K$. Then $x_{n_{k_l}}$ is a subsequence of the convergent sequence x_{n_k}, so it converges to the same element x.

Consider the subsequence $z_{n_{k_l}}$. We claim that $z_n \to z = (x, y)$. To this end, given $\epsilon > 0$, find L_1 such that $d(x_{n_{k_l}}, x) < \frac{\epsilon}{2}$ for $l \geq L_1$ and L_2 such that $d(y_{n_{k_l}}, y) < \frac{\epsilon}{2}$ for $l \geq L_2$, and set $L = \max\{L_1, L_2\}$. Then, for $l \geq L$ we have

$$d(z_{n_{k_l}}, z) = d(x_{n_{k_l}}, x) + d(y_{n_{k_l}}, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus z_n has a convergent subsequence. Since z_n was arbitrary, $K \times K$ is compact.

b

Let $D = \sup\{d(x, y) \mid x, y \in K\}$. By definition of the supremum we can for each n find $x_n, y_n \in K$ such that $d(x_n, y_n) > D - 1/n$. Let $z_n = (x_n, y_n)$, and find a convergent subsequence z_{n_k}, which we can do since $K \times K$ is compact. Let $z_{n_k} \to z = (x_0, y_0)$. Then $d(x_0, y_0) = \lim_{k\to\infty} d(x_{n_k}, y_{n_k}) = D$ since the metric $d\colon K \times K \to \mathbb{R}$ is continuous.

Problem 4

a

We first show that I is bounded. To this end, let $f, g \in I$. Then

$$\rho(f, g) = \sup_{x\in K} d(f(x), g(x)) \leq \sup_{x,y\in K} d(x, y) < \infty,$$

since we know from (3b) that the diameter of K is finite. Hence I is bounded.

To see that I is closed, assume that f_n is a sequence in I converging to some function f. For $x, y \in K$ and any n we then have

$$\begin{aligned}
|d\big(f(x), f(y)\big) - d(x,y)| &= |d\big(f(x), f(y)\big) - d\big(f_n(x), f_n(y)\big)| \\
&= |d\big(f(x), f(y)\big) - d\big(f(y), f_n(x)\big) \\
&\quad + d\big(f(y), f_n(x)\big) - d\big(f_n(x), f_n(y)\big)| \\
&= |d\big(f(x), f(y)\big) - d\big(f(y), f_n(x)\big) \\
&\quad + d\big(f(y), f_n(x)\big) - d\big(f_n(x), f_n(y)\big) \\
&\leq \big|d\big(f(x), f(y)\big) - d\big(f(y), f_n(x)\big)\big| \\
&\quad + \big|d\big(f(y), f_n(x)\big) - d\big(f_n(x), f_n(y)\big)\big| \\
&\leq d\big(f(x), f_n(x)\big) + d\big(f(y), f_n(y)\big).
\end{aligned}$$

Here the right-hand side converges to 0 as $n \to \infty$ while the left-hand side is a non-negative constant, so it must be zero. Hence we must have $d\big(f(x), f(y)\big) = d(x,y)$ for all $x, y \in K$, so $f \in I$. Hence I is closed.

b

By (a) the only thing missing is to show that I is equicontinuous.

Given $\epsilon > 0$, let $\delta = \epsilon$. For $f \in I$ and $x, y \in K$ with $d(x,y) < \delta$ we then have

$$d\big(f(x), f(y)\big) = d(x,y) < \delta = \epsilon,$$

as required. Hence I is equicontinuous.

By the Arzelà-Ascoli theorem I is compact.