# UNIVERSITY OF OSLO

## Faculty of mathematics and natural sciences

Exam in: MAT2400 — Real Analysis

Day of examination: Wednesday 14. June 2017

Examination hours: 09:00 – 13:00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

**For each question you may use results from previous questions, even if you have not answered them.**

We use the following notation:

$$\mathbb{R}_+ = \{x \in \mathbb{R} \ : \ x \geq 0\}. \tag{1}$$

Recall that if $(X, d_X)$ and $(Y, d_Y)$ are metric spaces, a map $u : X \to Y$ is said to be Lipschitz continuous iff there exists $M \in \mathbb{R}_+$ such that:

$$\forall x, y \in X \quad d_Y(u(x), u(y)) \leq M d_X(x, y). \tag{2}$$

In what follows, subsets of $\mathbb{R}$ and $\mathbb{C}$ are always equipped with the standard metric.

## Problem 1 (weight 20%)

Suppose that $u : \mathbb{R} \to \mathbb{R}$ is a continuous map, which has finite limits in both $-\infty$ and $+\infty$. That is, there are $a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that:

$$\lim_{x \to -\infty} u(x) = a \qquad \text{and} \qquad \lim_{x \to +\infty} u(x) = b. \tag{3}$$

### a (weight 10%)

Show that there exist $B \in \mathbb{R}_+$ and $C \in \mathbb{R}$, such that for all $x \in \mathbb{R}$, if $x \geq C$ then $|u(x)| \leq B$.

### b (weight 10%)

Show that $u$ is bounded - that is, show that there exists $M \in \mathbb{R}_+$ such that for all $x \in \mathbb{R}$ we have $|u(x)| \leq M$.

*(Continued on page 2.)*

## Problem 2 (weight 10%)

Let $X$ be a normed vector space. We suppose that $(u_n)_{n\in\mathbb{N}}$ is a sequence in $X$ that converges to a limit $\ell \in X$, that is:

$$\lim_{n\to\infty} u_n = \ell. \tag{4}$$

We define a new sequence $(v_n)_{n\in\mathbb{N}}$ by putting, for any $n \in \mathbb{N}$:

$$v_n = \frac{1}{n+1}\sum_{k=n}^{2n} u_k. \tag{5}$$

Show that $(v_n)_{n\in\mathbb{N}}$ converges to $\ell$, that is:

$$\lim_{n\to\infty} v_n = \ell. \tag{6}$$

## Problem 3 (weight 20%)

We let $\mathcal{X}$ denote the complex vector space of continuous maps from $[-\pi,\pi]$ to $\mathbb{C}$.

Recall that for any $u \in \mathcal{X}$, the Fourier coefficients of $u$ are the numbers $\alpha_k \in \mathbb{C}$ defined for $k \in \mathbb{Z}$ by:

$$\alpha_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)\exp(-ikx)\mathrm{d}x. \tag{7}$$

### a (weight 10%)

We suppose that $u$ is the restriction to $[-\pi,\pi]$ of a continuous $2\pi$-periodic map from $\mathbb{R}$ to $\mathbb{C}$, also denoted $u$. By a change of variables show that the Fourier coefficients of $u$, defined in (7), satisfy (for $k \neq 0$):

$$\alpha_k = -\frac{1}{2\pi}\int_{-\pi}^{\pi} u(x+\frac{\pi}{k})\exp(-ikx)\mathrm{d}x. \tag{8}$$

### b (weight 10%)

We suppose now that $u$ is the restriction to $[-\pi,\pi]$ of a Lipschitz continuous $2\pi$-periodic map from $\mathbb{R}$ to $\mathbb{C}$, also denoted $u$. Justify that we have (for $k \neq 0$):

$$\alpha_k = \frac{1}{4\pi}\int_{-\pi}^{\pi} (u(x) - u(x+\frac{\pi}{k}))\exp(-ikx)\mathrm{d}x. \tag{9}$$

Deduce that there exists $C \in \mathbb{R}_+$ such that for all $k \in \mathbb{Z}\setminus\{0\}$:

$$|\alpha_k| \leq \frac{C}{k}. \tag{10}$$

*(Continued on page 3.)*

## Problem 4 (weight 50%)

We let $\mathcal{X}$ denote the complex vector space of continuous maps from $[-\pi, \pi]$ to $\mathbb{C}$.

We let $\mathcal{Y}$ denote the subset of $\mathcal{X}$ consisting of Lipschitz continuous maps from $[-\pi, \pi]$ to $\mathbb{C}$.

### a (weight 10%)

Show that $\mathcal{Y}$ is a linear subspace of $\mathcal{X}$.

### b (weight 10%)

We define, for any $u \in \mathcal{Y}$, the following set of real numbers:

$$\mathrm{LIP}(u) = \{M \in \mathbb{R}_+ \ : \ \forall x, y \in [-\pi, \pi] \quad |u(x) - u(y)| \leq M|x - y|\}, \tag{11}$$

and the following real number:

$$\mathrm{Lip}(u) = \inf \mathrm{LIP}(u). \tag{12}$$

Show that for any $u \in \mathcal{Y}$, the set $\mathrm{LIP}(u)$ is closed in $\mathbb{R}$, and deduce that we have $\mathrm{Lip}(u) \in \mathrm{LIP}(u)$.

### c (weight 10%)

We define, for any $u \in \mathcal{Y}$:

$$\|u\|_{\mathcal{Y}} = |u(0)| + \mathrm{Lip}(u). \tag{13}$$

Show that $\| \cdot \|_{\mathcal{Y}}$ is a norm on $\mathcal{Y}$.

### d (weight 10%)

We define, for any $u \in \mathcal{X}$:

$$\|u\|_{\mathcal{X}} = \sup\{|u(x)| \ : \ x \in [-\pi, \pi]\}. \tag{14}$$

Recall that $\| \cdot \|_{\mathcal{X}}$ is a norm on $\mathcal{X}$.

Show that there exists a constant $C \in \mathbb{R}_+$ such that for all $u \in \mathcal{Y}$, $\|u\|_{\mathcal{X}} \leq C\|u\|_{\mathcal{Y}}$.

Which continuity property does this express?

### e (weight 10%)

Show that there does *not* exist a constant $C' \in \mathbb{R}_+$ such that for all $u \in \mathcal{Y}$, $\|u\|_{\mathcal{Y}} \leq C'\|u\|_{\mathcal{X}}$.

END